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THE OPTIMAL HOMOTOPY ASYMPTOTIC METHOD FOR SOLVING SOME STRONGLY NONLINEAR PROBLEMS

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Abstract: *In this paper an analytical technique namely OHAM is proposed to obtain analytical approximate solutions for strongly nonlinear problems. Complementary numerical solutions were obtained via a fourth-order Runge–Kutta method and an excellent agreement between the solutions obtained through OHAM and the numerical computations was observed, which demonstrate the reliability and efficiency of OHAM.*
Keywords: *Strongly nonlinear problems, Analytical approach, OHAM*

1. INTRODUCTION

Often dynamical problems lead to nonlinearity occurrence in the mathematical model describing the mechanical behavior. Sometimes, when a complicated and accurate dynamical model is employed, the motion of the system is described by strongly nonlinear differential equations, which are very difficult to solve through classic analytical methods.

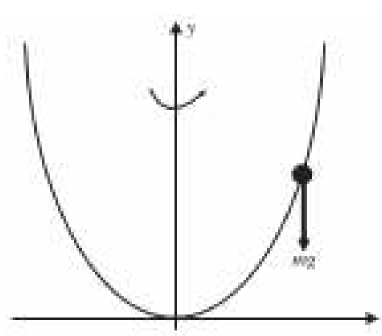
In order to overcome the shortcomings of classical methods, in the last years some new approximate analytical methods were developed, which are valid and work very well even in the absence of small parameters that is characteristic only for weakly nonlinear problems.

Among these new emergent methods, an important place was taken by some iterative methods, which are found to be very effective in many practical applications [1-7].

Beside these new iterative methods, some homotopy methods were developed in the last years [8-9], some of them having some limitations and some of them having some "open questions", which are still unsolved at this moment [8].

The newest homotopy method, proposed first by Marinca and Herişanu in 2007 is the Optimal Homotopy Asymptotic Method (OHAM) [10-13], which proved to be a very powerful analytical tool for strongly nonlinear problems arising in various fields of science and engineering.

In this paper, the Optimal Homotopy Asymptotic Method (OHAM) is applied to investigate the motion of a particle on a parabola which rotates with a constant angular velocity about the y-axis as shown in Fig.1.



. **Figure 1:** The particle on a rotating parabola

This kind of dynamic model is governed by the following nonlinear differential equation, mentioned by Nayfeh and Mook in [15]:

$$\left(1+4q^2X^2\right)\frac{d^2X}{dt^2}+\Lambda X+4q^2\left(\frac{dX}{dt}\right)^2X=0 \tag{1}$$

with the boundary conditions:

$$X(0)=a\ \ ,\ \ \frac{dX}{dt}(0)=0 \tag{2}$$

where q and Λ are known constants and need not to be small.

2. FUNDAMENTALS OF OHAM

The homotopy is a basic concept in topology and has been widely applied in developing some numerical algorithms.

Starting from the general given equation which describes a system oscillating with an unknown period T

$$\ddot{X}(t)+k^2X(t)=f\left(X(t),\dot{X}(t),\ddot{X}(t)\right) \tag{3}$$

where the dot denotes the derivative with respect to time, k is a constant, f is in general a nonlinear term and the initial conditions are:

$$X(0)=a\ \ ,\ \ \dot{X}(0)=0 \tag{4}$$

where a is the amplitude of the oscillations, it will be more convenient to switch to a scalar time $\tau=2\pi t/T=\Omega t$. Under the proposed transformation:

$$\tau=\Omega t\ \ ,\ \ X(t)=ax(\tau) \tag{5}$$

In these conditions the original Eq.(3) becomes:

$$\Omega^2x''(\tau)+k^2x(\tau)=\frac{f(ax(\tau),a\Omega x'(\tau),a\Omega^2x''(\tau))}{a} \tag{6}$$

while the initial conditions become:

$$x(0)=1\ \ ,\ \ x'(0)=0 \tag{7}$$

where the prime denotes derivative with respect to τ.

In order to solve the problem through OHAM, we construct a homotopy H(ϕ ,h):Rx[0,1] $\to$ R which satisfies:

$$(1-p)L(\phi(\tau,p))=h(\tau,p)N[\phi(\tau,p),\Omega(\lambda,p)]=0 \tag{8}$$

where L is a linear operator:

$$L(\phi(\tau,p))=\Omega_0^2\left[\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+\phi(\tau,p)\right] \tag{9}$$

while N is a nonlinear operator:

$$N[\phi(\tau,p),\Omega(\lambda,p)]=\Omega^2(\lambda,p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2}+(k^2+\lambda)\phi(\tau,p)-\frac{1}{a}f(a\phi(\tau,p),a\Omega(\lambda,p)\frac{\partial\phi(\tau,p)}{\partial\tau},a\Omega^2(\lambda,p)\frac{\partial^2\phi(\tau,p)}{\partial\tau^2})-p\lambda\phi(\tau,p) \tag{10}$$

where $p\in[0,1]$ is the embedding parameter, $h(\tau,p)$ is an auxiliary function so that $h(\tau,0)=0$, $h(\tau,p)\neq 0$ for $p\neq 0$, λ is an arbitrary parameter and Ω_0 will be given later. From Eqs.(4) and (5) the initial conditions become in this case:

$$\phi(0,p)=1\ \ ,\ \ \left.\frac{\partial\phi(\tau,p)}{\partial\tau}\right|_{\tau=0}=0 \tag{11}$$

Obviously when p=0
and p=1 it holds:

$$\phi(\tau,0)=x_0(\tau)\ ,\ \phi(\tau,1)=x(\tau) \tag{12}$$

where $x_0(\tau)$ is an initial approximation of $x(\tau)$ and when p=1 it holds:

$$\Omega(0)=\Omega_0\ \ ,\ \ \Omega(1)=\Omega \tag{13}$$

Thus, as the embedding parameter p increases from 0 to 1, $\phi(\tau,p)$ varies (or deforms) from the initial approximation $x_0(\tau)$ to the solution $x(\tau)$ *of the initial equation*, so does $\Omega(p)$ from the initial guess Ω_0 to the exact frequency Ω.

Expansions of $\phi(\tau,p)$ and $\Omega(p)$ in series with respect to the embedding parameter p lead to:

$$\phi(\tau,p)=x_0(\tau)+px_1(\tau)+p^2x_2(\tau)+..... \tag{14}$$

$$\Omega(p)=\Omega_0+p\Omega_1+p^2\Omega_2+...... \tag{15}$$

where series (14) and (15) contain the auxiliary function h(τ,p) which will ensure their convergence.

An appropriate expression for the initial approximation $x_0(\tau)$ and for the auxiliary function h(τ,p), which ensure the convergence of the above series at p=1, leads to obtaining the m th-order approximate solutions given by

$$\bar{x}(\tau) \approx x_0(\tau) + x_1(\tau) + ... + x_m(\tau) \quad \textbf{(16)}$$

$$\bar{\Omega} = \Omega_0 + \Omega_1 + ...\Omega_m \quad \textbf{(17)}$$

The auxiliary function h(τ,p) could be chosen of the form:

$$h(\tau,p) = pK_1 + p^2K_2 + ... + p^mK_m(\tau) \quad \textbf{(18)}$$

where K_1, $K_2, ..., K_{m-1}$ can be constants which are later optimally determined and the last value $K_m(\tau)$ can be a function depending on the variable τ.

Substitution of Eqs.(14) and (15) into Eq.(10) yields:

$$N(\phi,\Omega) = N_0(x_0,\Omega_0,a,\lambda) + pN_1(x_0,x_1,\Omega_0,\Omega_1,a,\lambda) + p^2N_2(x_0,x_1,x_2,\Omega_0,\Omega_1,a,\lambda) + ... \quad \textbf{(19)}$$

If we substitute Eqs.(19) and (18) into Eq.(8) and we equate to zero the coefficients of the same powers of p, we obtain the following linear equations:

$$L(x_0) = 0 \ , \ x_0(0) = 1 \ , \ x'(0) = 0 \quad \textbf{(20)}$$

$$L(x_i) - L(x_{i-1}) - \sum_{j=1}^{i} K_j N_{i-j}(x_0,x_1,...,x_{i-j},\Omega_0,\Omega_1,...,\Omega_{i-j},a,\lambda) = 0 \ , \ x_i(0) = x_i'(0) = 0 \ , \ i = 1,2,...,m-1$$

$$L(x_m) - L(x_{m-1}) - \sum_{j=1}^{m-1} K_j N_{m-1-j} - K_m(\tau)N_0 = 0 \ , \ x_m(0) = x_m'(0) = 0 \quad \textbf{(21)}$$

Note that Ω_k can be determined avoiding the presence of secular terms in the Eq.(21).

The frequency Ω depends upon the arbitrary parameter λ and we apply the so-called "principle of minimal sensitivity" in order to fix the value of λ imposing that:

$$\frac{d\Omega}{d\lambda} = 0 \quad \textbf{(22)}$$

At this moment, the m th-order approximation given by Eq.(16) depends on the parameters (functions) K_1, $K_2,...,K_m$. The constants K_1, K_2, ... K_{m-1} and those constants which eventually appear in the expression of $K_m(\tau)$, can be identified via various methods, such as the least square method, the Galerkin method, the collocation method or by minimizing the square residual error minimizing the functional

$$J(C_1,C_2,...,C_m) = \int_a^b R^2(\tau,C_1,C_2,...,C_m)d\tau \quad \textbf{(23)}$$

where a and b are two values, from the domain of the given problem. The unknown constants C_i (i=1,2,…m) can be optimally identified from the conditions

$$\frac{\partial J}{\partial C_1} = \frac{\partial J}{\partial C_2} = ... = \frac{\partial J}{\partial C_m} = 0 \quad \textbf{(24)}$$

This procedure contains the auxiliary function $h(\tau,p)$, which provides us with a simple way to adjust and control the convergence of solution. It is very important to properly choose the last function $K_m(\tau)$, which appears in the m th-order approximation (16).

Unlike other homotopy methods, such as HAM or HPM, in the proposed procedure (OHAM) the construction of homotopy is quite different. In the frame of OHAM the linear operator L is well defined by Eq.(9) and the initial approximation is rigorously determined from Eq.(20), while in other homotopy procedures such as HAM these ones are arbitrarily chosen. Instead of an infinite series, as is the case of HAM [8], the OHAM searches for only a few terms (mostly two or three terms). The way to ensure the convergence in OHAM is quite different and more rigorous. Unlike other homotopy procedures [8,9], OHAM ensure a very rapid convergence since it always needs only two iterations for achieving a very accurate solution. This is in fact the true power of the method. OHAM does not need a recurrence formula as other homotopy procedures such as HAM does.

OHAM is an iterative procedure which converges to the exact solution after only two iterations. Iterations are performed in a very simple manner by identifying some coefficients. OHAM does not need high-order approximations, as HAM does. OHAM does not use the rules established in the frame of HAM, it is a self-sustained method which has no "open questions" as other homotopy procedures [8]. OHAM does not need the restrictive condition A(1)=1 as HAM does. Finally, OHAM provides an analytic solution for complicated nonlinear problems on only two rows, unlike other homotopy procedures which need few pages to express an analytic solution [8].

3. TESTING EXAMPLE

The procedure described above will be tested on the motion of a particle on a rotating parabola, described by Eq.(1). Under the transformations (5), Eq.(23) and Eq.(24) become:

$$\Omega^2x'' + \omega_0^2x + 4q^2a^2(x^2x'' + x\dot{x}^2) = 0 \quad \textbf{(25)}$$

respectively

$$x(0)=1 \quad , \quad x'(0)=0 \tag{26}$$

where $\Lambda = \omega_0^2$ and prime denotes differentiation with respect to τ.

The linear and nonlinear operators (9) and (10) will be:

$$L(\phi(\tau,p)) = \Omega_0^2\left[\phi''(\tau,p)+\phi(\tau,p)\right] \tag{27}$$

$$\begin{aligned} N\left[\phi(\tau,p),\Omega(\tau,p)\right] &= \Omega^2(p)\phi''(\tau,p)+(\omega_0^2+\lambda)\phi(\tau,p)+ \\ &+4q^2a^2\left[\phi^2(\tau,p)\phi''(\tau,p)+\phi(\tau,p)\phi'^2(\tau,p)\right]-p\lambda\phi(\tau,p) \end{aligned} \tag{28}$$

where ϕ and Ω are given by Eqs.(14) and (15) respectively and λ is an unknown parameter. From Eqs.(20) and (21), (m=2), we obtain the following three equations:

$$\Omega_0^2(x_0''+x_0)=0 \quad , \quad x_0(0)=1 \quad , \quad x'(0)=0 \tag{29}$$

$$\begin{aligned} &\Omega_0^2(x_1''+x_1)-\Omega_0^2(x_0''+x_0)- \\ &-K_1\left[\Omega_0^2x_0''+(\omega_0^2+\lambda)x_0+4q^2a^2\Omega_0^2(x_0x_0''+x_0'^2x_0)\right]=0 \quad , \quad x_1(0)=x_1'(0)=0 \end{aligned} \tag{30}$$

$$\begin{aligned} &\Omega_0^2(x_2''+x_2)-\Omega_0^2(x_1''+x_1)-K_1\{2\Omega_0\Omega_1x_0''+\Omega_0^2x_1''+(\omega_0^2+\lambda)x_1+4q^2a^2[\Omega_0^2(2x_0x_0''x_1+ \\ &+x_0^2x_1''+2x_0x_0'x_1'+2x_0x_0'x_1'+x_0'^2x_1)+2\Omega_0\Omega_1(x_0^2x_0''+x_0'^2x_0)]-\lambda x_0\}-K_2(\tau)[\Omega_0^2x_0''+ \\ &+(\omega_0^2+\lambda)x_0+4q^2a^2\Omega_0^2(x_0^2x_0''+x_0'^2x_0)]=0 \quad , \quad x_2(0)=x_2'(0)=0 \end{aligned} \tag{31}$$

Obviously Eq.(29) has the solution:

$$x_0(\tau)=\cos\tau \tag{32}$$

which will be the initial approximation. Substituting this result into Eq.(30) and assuming that $K_1=C_1$=constant, one obtain the following equation:

$$\Omega_0^2(x_1''+x_1)-C_1[(\omega_0^2+\lambda-\Omega_0^2-2q^2a^2\Omega_0^2)\cos\tau-2q^2a^2\Omega_0^2\cos3\tau]=0 \quad , \quad x_1(0)=x_1'(0)=0 \tag{33}$$

where C_1 is an unknown constant at this moment.

In order to avoid the presence of a secular term it must:

$$\Omega_0^2=\frac{\omega_0^2+\lambda}{1+2q^2a^2} \tag{34}$$

With this requirement, the solution of Eq.(33) is:

$$x_1(\tau)=\frac{1}{4}C_1q^2a^2(\cos3\tau-\cos\tau) \tag{35}$$

Substituting Eqs.(32), (34) and (35) into Eq.(31), one obtain the equation in x_2:

$$\begin{aligned} &\Omega_0^2(x_2''+x_2)+\frac{2C_1q^2a^2(\omega_0^2+\lambda)}{1+2q^2a^2}\cos3\tau+C_1\left\{\left[\frac{C_1q^4a^4(\omega_0^2+\lambda)}{2(1+2q^2a^2)}+2\Omega_0\Omega_1(1+2q^2a^2)+\lambda\right]\cos\tau+\right. \\ &+\left.\left[\frac{(\omega_0^2+\lambda)C_1q^2a^2(3q^2a^2+16)}{2(1+2q^2a^2)}+2\Omega_0\Omega_1q^2a^2\right]\cos3\tau+\frac{9C_1q^4a^4(\omega_0^2+\lambda)}{2(1+2q^2a^2)}\cos5\tau\right\}+ \\ &+K_2(\tau)\left[\frac{2q^2a^2(\omega_0^2+\lambda)}{1+2q^2a^2}\cos3\tau\right]=0 \quad , \quad x_2(0)=x_2'(0)=0 \end{aligned} \tag{36}$$

No secular term in $x_2(\tau)$ requires that

$$2\Omega_0\Omega_1=-\frac{\lambda}{1+2q^2a^2}-\frac{C_1q^4a^4(\omega_0^2+\lambda)}{2(1+2q^2a^2)^2} \tag{37}$$

From Eqs.(37) and (17), one obtain the frequency in the form:

$$\Omega=\Omega_0-\frac{\lambda}{\Omega_0(1+2q^2a^2)}-\frac{C_1q^4a^4\Omega_0}{4(1+2q^2a^2)} \tag{38}$$

where Ω_0 is given by Eq.(34).

The parameter λ can be determined applying the so-called "principle of minimal sensitivity" (22) and thus we obtain

$$\lambda=\frac{C_1\omega_0q^4a^4}{2+4q^2a^2-C_1q^4a^4} \tag{39}$$

Substitution of this result into Eq.(38) lead to:

$$\Omega=\frac{\omega_0}{1+2q^2a^2}\sqrt{1+2q^2a^2-\frac{1}{2}C_1q^4a^4} \tag{40}$$

Substituting Eqs.(38), (39) and (40) into Eq.(36), one obtain:

$$x_2'' + x_2 + 2C_1 q^2 a^2 \cos 3\tau + \frac{C_1^2 q^2 a^2 (5q^4 a^4 + 7q^2 a^2 + 2)}{1 + 2q^2 a^2} \cos 3\tau +$$

$$+ \frac{9}{2} C_1^2 q^4 a^4 \cos 5\tau + 2K_2(\tau) q^2 a^2 \cos 3\tau = 0 \quad \textbf{(41)}$$

$$x_2(0) = x_2'(0) = 0$$

Considering the function K_2 of the simplest form:

$$K_2(\tau) = C_2' \quad \textbf{(42)}$$

where C_2' is a constant and substituting Eq.(42) into Eq.(41), we obtain the equation in x_2:

$$x_2'' + x_2 + \left[2(C_1 + C_2')q^2 a^2 + \frac{C_1^2 q^2 a^2 (5q^4 a^4 + 7q^2 a^2 + 2}{1 + 2q^2 a^2} \right] \cos 3\tau + \frac{9}{2} C_1^2 q^4 a^4 \cos 5\tau \quad \textbf{(43)}$$

$$x_2(0) = x_2'(0) = 0$$

The solution of Eq.(43) becomes:

$$x_2(\tau) = \left[\frac{C_1 + C_2'}{4} + \frac{C_1^2 q^2 a^2 (5q^4 a^4 + 7q^2 a^2 + 2}{8(1 + 2q^2 a^2)} \right] (\cos 3\tau - \cos \tau) + \frac{3}{16} C_1^2 q^4 a^4 (\cos 5\tau - \cos \tau) \quad \textbf{(44)}$$

The second order approximate solution will be

$$\bar{x}(\tau) = x_0(\tau) + x_1(\tau) + x_2(\tau) \quad \textbf{(45)}$$

where x_0, x_1 and x_2 are given by Eqs. (32), (35) and (44). Using the transformations (5), the second order approximate solution of Eq.(1) becomes:

$$\bar{x}(t) = A\cos \Omega t + B\cos 3\Omega t + C\cos 5\Omega t \quad \textbf{(46)}$$

where Ω is given by Eq.(40) and

$$A = a - \frac{2C_1 + C_2'}{4} q^2 a^3 - \frac{C_1^2 q^2 a^3 (16q^4 a^4 + 17q^2 a^2 + 4)}{16(1 + 2q^2 a^2)}$$

$$B = \frac{2C_1 + C_2'}{4} q^2 a^3 + \frac{C_1^2 q^2 a^3 (5q^4 a^4 + 7q^2 a^2 + 2)}{8(1 + 2q^2 a^2)} \quad \textbf{(47)}$$

$$C = \frac{3}{16} C_1^2 q^4 a^5$$

The constants C_i are obtained in this case using the least square method, as follows:

For q=1, it is obtained:

$$C_1 = -0.401483291 \ , \ C_2' = -0.065781508 \quad \textbf{(48)}$$

The second order approximate solution (46) becomes in this case:

$$\bar{x}(t) = 1.092937297\cos \Omega t - 0.123160203\cos 3\Omega t + 0.030222906\cos 5\Omega t \quad \textbf{(49)}$$

where Ω is obtained from Eq.(40): Ω=0.596353888.

Fig.1 shows the comparison between the present solutions and the numerical integration results obtained by a fourth-order Runge-Kutta method.

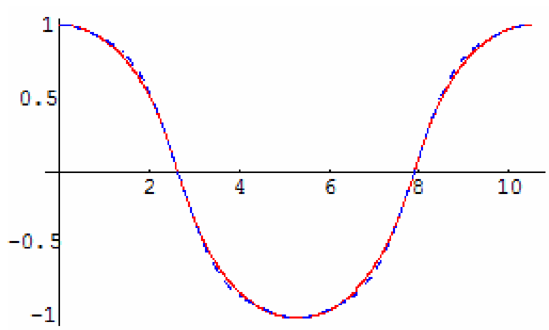


Figure 2: Comparison between the approximate solution - Eq.(49) and numerical results from Eq.(1) for $\Lambda = \omega_0 = a = q = 1$

______ numerical simulation _ _ _ _ approximate solution

It can be seen from Fig.2 that the solutions obtained by means of OHAM are nearly identical with the solutions obtained numerically by means of a fourth-order Runge-Kutta method.

3. CONCLUSION

In this paper an analytical technique namely OHAM is proposed to obtain analytical approximate solutions for some strongly nonlinear oscillations. The validity of the method is illustrated on the motion of a particle on a rotating parabola. Complementary numerical solutions were obtained via a fourth-order Runge–Kutta method and an excellent agreement between the solutions obtained through OHAM and the numerical computations was observed, which demonstrate the reliability and efficiency of OHAM. This method is valid not only for small, but also for large parameters and provides a convenient and rigorous way to control the convergence of approximate solution through the auxiliary functions $h(\tau,p)$ involving a number of constants C_i which are optimally determined..

This method is quite different from classical HAM, especially referring to the parameter λ (determined using the principle of minimal sensitivity), the auxiliary function $h(\tau,p)$, the linear operator *L (*unlike HAM the linear operator and the initial approximation are not arbitrary chosen) and the presence of some constants C_1, C_2, … which ensure a fast convergence of the method.

Unlike HAM which needs recurrence formulas, OHAM is an iterative procedure and iterations are performed in a very simple manner by identifying some coefficients and therefore very good approximations are obtained in few terms. Actually the capital strength of OHAM is its fast convergence, since after only two iterations it converges to the exact solution.

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