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ON THE NUMERICAL ANALYSIS OF COMPOSITE MATERIAL

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Abstract: *In this paper we analyze laminate composite materials by numerical methods. We write a Matlab program that assists the user to find out the ABBD stiffness matrix of a laminate composite. The main objective of the paper is to show the advantages and ease use of Matlab software in composite materials analysis. To demonstrate the capability of the program a lot of numerical examples was presented.*
Keywords: *composite laminate, stiffness matrix, constitutive equations, MATLAB.*

1. INTRODUCTION

Numerous papers have been published on the analysis and design composite material. Because composite materials are produced in many combinations and forms the designer engineer must consider many design cases.
In study of structural response composite material is often analyzed by analytical methods and by numerical methods [2], [3], [8], [9]. This requires a large amount of calculations that depend on many parameters, Calculations of macro-mechanical proprieties and calculations of the constitutive equations involve many matrix manipulations. The manual calculations would take long time. Solutions was to write computers programs. Numerical computing package MATLAB was used as a basis for the programs because the facilities offered in matrix computations [1], [4], [6], and [7]. The main objective of this paper is to show the advantages of using MATLAB to analysis of composite material.
We illustrate these features by analyzing orthotropic laminate composite material. Fiber-reinforced composites are analyzed by two-dimensional theories. The Kirchoff –Love hypothesis is used. The objective is to write a program to determine the laminate constitutive equations for multi-layered composites. This approach follows the conventional methods for designing composite structures. When an orthotropic materials is in a plane stress state the relationship between the stresses and strain involve the four elastic constants E_1, E_2, ν_{12} ,and G_{12} . In section 2 we present the characteristics of unidirectional composite material as functions of the characteristic of the fibres and the matrix. In section 3 we will review basis assumptions for study the constitutive relations for fiber-reinforced composite and calculate stiffness matrices for laminate, ABBD matrix. In sections 4 details for computational approach have been outlined. How to coding in MATLAB software has been outlined step-by-step procedure. Numerical applications are presented in section 5. In section 6 we present the conclusions.

2. EVALUATION OF ELASTIC CONSTANTS

Stiffness and strength is the basic concept for underlying the mechanics of fiber-reinforced advanced composite materials. This aspect of composite materials technology is sometimes terms micromechanics, because it deals with the relations between macroscopic engineering properties and the microscopic distribution of the material's constituents, namely the volume fraction of fiber. The purpose of this section is to predict the material constants (also called elastic constants) of a composite material by studying the micromechanics of the problem, i.e. by studying how the matrix and fibers interact.
There are three different approaches that are used to determine the elastic constants for the composite material based on micromechanics. These three approaches are [2], [3]:
1. Using numerical models such as the finite element method.
2. Using models based on the theory of elasticity.
3. Using rule-of-mixtures models based on a strength-of-materials approach.

Terminology used in micromechanics:

- E_v , E_m -Young's modulus of fiber and matrix
- G_f , G_m -Shear modulus of fiber and matrix
- ν_f , ν_m -Young's modulus of fiber and matrix
- V_f , V_m -Volum fraction of fiber and matrix;

There are four elastic constant of a unidirectional lamina:

- Longitudinal Young's modulus: E_1
- Transverse Young's modulus : E_2
- Major Poisson's ratio ν_{12}
- In-plane shear modulus: G_{12}

Using the strength-of-materials approach and the simple rule of mixtures, we have the following relations for the elastic constants of the composite material in local coordinates systems, [1], [5].

Longitudinal Young's modulus in the 1-direction (the longitudinal stiffness):

$$E_1 = E_f V_f + E_m V_m \qquad (1)$$

Transvers Young's modulus in the 2-direction (also called the transverse stiffness):

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m} \qquad (2)$$

Poisson's ratio ν_{12} in the 1-2 plane:

$$\nu_{12} = \nu_f V_f + \nu_m V_m \qquad (3)$$

where ν_f and ν_m are Poisson's ratios for the fiber and matrix, respectively.

Shear modulus in the 1-2 plane G_{12}:

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m} \qquad (4)$$

While the simple rule-of-mixtures models used above give accurate results for E_1 and ν_{12}, the results obtained for E_2 and G_{12} do not agree well with finite element analysis and elasticity theory results. Therefore, we need to modify the simple rule-of-mixtures models shown above.

In Matlab file we using the elasticity solution give the following formula for G_{12}:

$$G_{12} = \frac{G_m(G_f + G_m + (G_f - G_m)V_f)}{G_f + G_m - (G_f - G_m)V_f} \qquad (5)$$

3. THE STRESS-STRAIN RELATION

This approach follows the conventional methods for designing composite structures. Equations relating the stresses and strains have been developed and are available from various tests [2], [3], [5], and [8]. Fiber-reinforced composite are analyze by two-dimensional theories. The Kirchoff-Love hypothesis is used. Using the assumption of plane stress, the relationship between the stress-strain involve the four elastic constants: E_1, E_2, ν_{12}, G_{12}, determined in section 2. It is seen that the stress-strain relations for a single 2D orthotropic lamina are:

$$\{\varepsilon\} = [S]\{\sigma\} \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{\nu_{12}}{E_1} & 0 \\ \frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \qquad (6)$$

$$\{\sigma\} = [Q]\{\varepsilon\} \quad \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{Bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \gamma_{12} \end{Bmatrix} \qquad (7)$$

where Q_{ij} are the reduced stiffness constants.

The 1-2 co-ordinate system can be considered to be local co-ordinates based on the fibre direction. However this system is inadequate as fibres can be placed at various angles with respect to each other and the structure. Therefore a new co-ordinate system needs to be defined that takes into account the angle the fibre makes with its surroundings. This new system is referred to as global co-ordinates (*x-y* system) and is related to the local co-ordinates (1-2 system) by the angle θ, Fig.1.
To find the stress and strain in the (x,y,z) global coordinate system a simple rotational transformation is need. The transformation relation from local stresses to the global stresses is:

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & -n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{Bmatrix} \quad (9)$$

where m= cosθ; n = sinθ.
Similar transformation relations hold for the strains.
The transformed reduced stiffness matrix and the stress-strain relations to a global (*x, y, z*) coordinates are:

$$[\overline{Q}] = [T]^{-1}[Q][T] \; ; \quad \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (10)$$

Note that the following relations hold $[\overline{Q}] = [\overline{S}]^{-1}$; $[\overline{S}] = [\overline{Q}]^{-1}$.
Equations (1) - (10) are used to calculate the stresses and strain for a single layer.
Fiber-reinforced materials consist usually of multiple layers of material to form a laminate.
Consider a plate of total thickness h composed of N orthotropic layers with the principal material coordinates oriented at angles $\theta_1, \theta_2, ..., \theta_N$.
We omit the details, and summarize the laminate constitutive equations for multi-layered composites. The laminate constitutive equations relate the force and moment resultants (N_i , M_i) to the vector $\{\varepsilon_x^0 \quad \varepsilon_y^0 \quad \gamma_{xy}^0\}^T$ of the mid- plane strains and to the vector of the mid- plane curvatures $\{\kappa_x^0 \quad \kappa_y^0 \quad \kappa_{xy}^0\}$.
The constitutive equations are:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ \kappa^0 \end{Bmatrix} \quad (11)$$

where the 6x6 laminate matrix consisting of the components A_{ij}, B_{ij}, D_{ij} (i, j = 1,2,6) is laminate stiffness matrix, also called ABBD matrix.
The sub-matrix [A] is called the extensional stiffness matrix, sub-matrix [B] is called the coupling stiffness matrix and sub-matrix [D] is called the bending stiffness matrix.
The stiffness coefficients are determined by the following equations:

$$A_{ij} = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (h_k - h_{k-1}) = \sum_{k=1}^{N} (\overline{Q}_{ij})_k t_k \quad (12)$$

$$B_{ij} = \frac{1}{2} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (h_k^2 - h_{k-1}^2) = \sum_{k=1}^{N} (\overline{Q}_{ij})_k t_k \overline{h}_k \quad (13)$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{N} (\overline{Q}_{ij})_k (h_k^3 - h_{k-1}^3) = \sum_{k=1}^{N} (\overline{Q}_{ij})_k (t_k \overline{h}_k^2 + \frac{t_k^3}{12}) \quad (14)$$

where N is the number of layers, $(\overline{Q}_{ij})_k$ are the elements of $[\overline{Q}]$ matrix for he k-th layer, h_k is the distance from the mid-plane of laminates, Fig. 2.

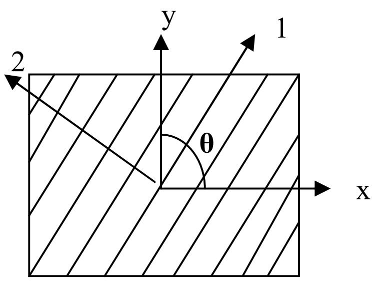

Figure 1: Local / global coordinate systems

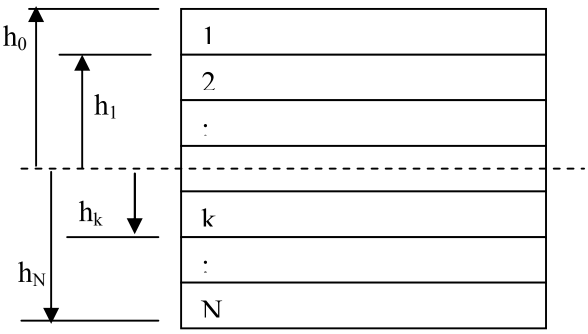

Figure 2: Cross-section input data

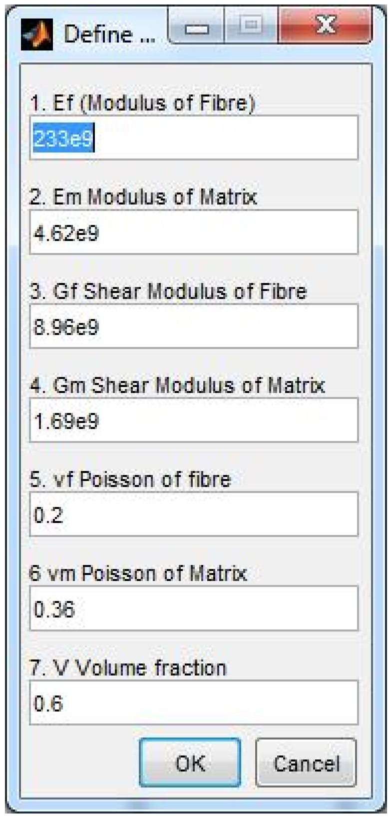

Figure 3: Interactive interface input data

4. MATLAB PROGRAM

Manual calculations would take a long period therefore the laminate analysis outlined above has been implemented in a MATLAB code. This program finds the overall laminate proprieties.
The outputs of the program are $[A]$, $[B]$ and $[D]$ matrices.
If we are going to make a laminated structure we must know the material proprieties, number of fiber layers, the thickness of each ply, the angle of each ply (in degrees) from the top of layer down, these are the inputs. A user interactive interface input data was made, Fig. 3.
In the Matlab script the calculations are done in the following step:
1. Micromechanics:

- enter the fiber and matrix proprieties (from the user interface) $E_f, E_m, G_f, G_m, V_f, V_m$;
- calculate ply elastic proprieties E_1, E_2, ν_{12}, G_{12} using Equations (1) - (5);

2. Enter the laminate characteristic:

- angle of fibers and thickness of each layer;
- number of layers;
- material proprieties of each layer.
- total height of the laminate

3. Calculate the stiffness matrix $[Q]$ for each layer using Eq. (7).
4. Calculate the transformation matrix $[T]$ for each layer.
5. Calculate the transformed reduced stiffness matrix $[\overline{Q}]$ for each layer using Eq. (10).
6. Calculate $[A]$, $[B]$ and $[D]$ matrices via Eq. (12)-(14).

This approach follows the commonly found methods laid out in the various texts [2], [3], [5], and [9].
Program consist in a main script file and various functions. These Matlab functions correspond to the steps of the program described above.

4. NUMERICAL EXAMPLE

The following example is taken to demonstrate the capability of the program. This example demonstrates the use of Matlab code in determining the stiffness of a four-ply layup. The material chosen for the composite plies was Graphite/Epoxy (Gr/E). This material consists of carbon (graphite) fibers in a standard epoxy matrix. Table 1 summarizes the composite properties. Values displayed in the table are standard values for Gr/E composite with a 0.55 volume fraction.
The ply angle orientations $[0 \backslash 90 \backslash 90 \backslash 0]$ was considered.

Table 1: Material Proprieties

Material	E_1	E_2	G_{12}	ν_{12}	t
Gr/E Composite	155	12.10	4.40	0.248	0.5

The outputs of the program for 4-ply systems are $[A]$, $[B]$ and $[D]$ stiffness matrices, listed below:

$$A=\begin{bmatrix} 1.6791e+011 & 6.0306e+009 & -2.1009e-008 \\ 6.0306e+009 & 1.6791e+011 & 8.8133e-006 \\ -2.1009e-008 & 8.8133e-006 & 8.8e+009 \end{bmatrix}$$

$$D=\begin{bmatrix} 9.1866e+010 & 2.0102e+009 & -1.7507e-009 \\ 2.0102e+009 & 2.0071e+010 & 7.3444e-007 \\ -1.7507e-009 & 7.3444e-007 & 2.9333e+009 \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[A\ B;\ B\ D]=\begin{bmatrix} A & B \\ B & D \end{bmatrix}=$$

$$\begin{bmatrix} 1.6791e+011 & 6.0306e+009 & -2.1009e-008 & 0 & 0 & 0 \\ 6.0306e+009 & 1.6791e+011 & 8.8133e-006 & 0 & 0 & 0 \\ -2.1009e-008 & 8.8133e-006 & 8.8e+009 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9.1866e+010 & 2.0102e+009 & -1.7507e-009 \\ 0 & 0 & 0 & 2.0102e+009 & 2.0071e+010 & 7.3444e-007 \\ 0 & 0 & 0 & -1.7507e-009 & 7.3444e-007 & 2.9333e+009 \end{bmatrix}$$

In order to ensure the validity of the program, the result was compared with those reported by Fiber Reinforced Composites (FRC) calculator in the website efunda.com [9], listed below:
The extensional stiffness matrix [*A*]:

$$[\boldsymbol{A}] = \sum_{k=1}^{N} \left(\overline{C}_{ij}\right)_k (z_k - z_{k-1}) = \sum_{k=1}^{N} \left(\overline{C}_{ij}\right)_k t_k$$

$$= \begin{vmatrix} 167.9 & 6.031 & 0 \\ 6.031 & 167.9 & 0 \\ 0 & 0 & 8.800 \end{vmatrix} \text{ GPa-mm}$$

The bending stiffness matrix [*D*]:

$$[\boldsymbol{D}] = \frac{1}{3}\sum_{k=1}^{N} \left(\overline{C}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right) = \sum_{k=1}^{N} \left(\overline{C}_{ij}\right)_k \left(t_k \bar{z}_k^2 + \frac{t_k^3}{12}\right)$$

	91.87	2.010	0	
=	2.010	20.07	0	GPa-mm^3
	0	0	2.933	

5. CONCLUSIONS

The objective of this paper was to show the advantages of using MATLAB software to study various laminates composite. In this work, go through the first stage of analysis of composite materials, which consist of stiffness matrix calculations. The program written in MATLAB allows the analysis of different laminates, by introducing different material proprieties and inner ply orientations from interface input. The input are the material properties, number of fiber layers, and thickness and fibre orientation of each layer. The output of the program are $[A]$, $[B]$ and $[D]$ stiffness matrices. The use of the program will greatly reduce the analysis and design time of laminate fiber – reinforced composite.

6. REFERENCES

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