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SPECTRAL RESPONSE FOR THE DUFFING OSCILLATOR WITH NON-LINEAR ELASTIC CHARACTERISTIC

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Abstract: *The statistical linearization techniques considered in this paper are all based on the concept of using an equivalent linear system to compute the necessary expectation which occur in the expressions for the equivalent linear parameters. This article explains the method of statistical linearization and its use in solving random vibration problems. Numerous examples, drawn from a wide variety of engineering problems, offer a comprehensive view of the methods practical applications. This paper include general equations of motion for the Duffing oscillator and the representation of non-linearityes, probability theory, elements of linear random vibration theory and statistical linearization for simple systems with stationary response oscillator. This approximate representation of the system leads to estimates of the response spectrum that agree extremely well with those obtained by direct numerical simulation of the governing equation.*
Keyword: *statistical linearization, random vibration, power spectral density, response, equivalent linearization.*

1. SYSTEM MODEL

Nonlinear, dynamic systems subject to random excitations are frequently met in engineering practice. The source of randomness can vary from surface randomness in vehicle motion and environmental changes, such as earthquakes or wind exciting high rise buildings or wave motions at sea exciting ofshore structures or ships, to electric or acoustic noise exciting mechanical structures. The research goals are, firstly, the computation of stochastic, nonlinear response characteristics (with accuracy and efficiency as important criteria) and, secondly, the investigation and thorough understanding of stochastic, nonlinear response phenomena. The desire to compute response characteristics, such as the statistical moments and the power spectral density of the response of these systems, leads to the development of methods that can be used to approximate this response. The excitations, that will be studied, are stationary, Gaussian processes. We present a method for estimating the power spectral density of the stationary response of oscillator with a nonlinear restoring force under external stochastic wide-band excitation. An equivalent linear system is derived, from which the power spectral density is deduced. The method of the stochastic equivalent linearization is based on the idea that a nonlinear system may be replaced by a linear system by minimizing the mean square error of the two systems. Difficulties inherent in handling nonlinear random vibration problems are well known. While some particular exact solutions are available for specific systems under white noise excitations, many practical problems have been handled by approximate approaches such as linearization and closure assumptions. Even in these approximate procedures estimation of non-Gaussian characteristics of the responses, joint density functions and PSD functions pose difficulties.The most commonly applied and convenient procedure uses Yingfang, L. Zhao, Q. Chen [2] suggestion to estimate the linearization coefficients in context with Wen's introduction of an analytical expression for the restoring force. This method has seen the broadest application because of their ability to accurately capture the response statistics over a wide range of response levels while maintaining relatively light computational burden. The method will be briefly discussed in the following sections To illustrate the procedure of equivalent linearization theory, let us consider the following oscillator with a nonlinear damping force component and the nonlinear elastic characteristic. The ordinary differential equation reduced of the motion can be written as:

$$\ddot{\eta}(t) + 2\xi pc[\dot{\eta}(t) + \varepsilon\phi_1 \dot{\eta}^3(t) + \varepsilon\phi_2 \dot{\eta}^5(t)] + p^2 \eta(t) + p^2 \alpha \eta^3(t) = f(t) \qquad (1)$$

where $\eta(t)$ is the displacement response of the system, c is the viscous damping coefficient, ξ is the critical damping factor, p is the undamped natural frequency, for the linear system, α is the nonlinear factor to control the type and degree of nonlinearity in the system, ϕ_1 and ϕ_2 - nonlinearity correction factor of damping feature.

The equation of motion can be rewritten as:

$$\ddot{\eta}(t)+h(\eta(t),\dot{\eta}(t))=f(t). \tag{2}$$

The linearized [2] equation is of the form:

$$\ddot{\eta}(t)+\beta_{ech}\dot{\eta}(t)+\gamma_{ech}\eta(t)=f(t). \tag{3}$$

The difference [2,3] between the nonlinear stiffness and linear stiffness terms is

$$\begin{aligned}\varepsilon &= h(\eta(t),\dot{\eta}(t))-\beta_{ech}\dot{\eta}(t)-\gamma_{ech}\eta(t)=2\xi pc[\dot{\eta}(t)+\varepsilon\phi_1\dot{\eta}^3(t)\\ &+\varepsilon\phi_2\dot{\eta}^5(t)]+p^2\eta(t)+p^2\alpha\eta^3(t)-\beta_{ech}\dot{\eta}(t)-\gamma_{ech}\eta(t)\end{aligned} \tag{4}$$

The value of parameter β_{ech} and γ_{ech} can be obtained by minimizing the expectation of the square error

$$\frac{\partial}{\partial\,\beta_{ech}}E[\varepsilon^2]=0 \tag{5}$$

and

$$\frac{\partial}{\partial\,\gamma_{ech}}E[\varepsilon^2]=0. \tag{6}$$

Because

$$E\{\varepsilon^2\}=E\{h^2\}+\beta_{ech}{}^2E\{\dot{\eta}^2\}+\gamma_{ech}{}^2E\{\eta^2\}-2\beta_{ech}E\{\dot{\eta}h\}+2\beta_{ech}\gamma_{ech}E\{\eta\dot{\eta}\}-2\gamma_{ech}E\{\eta h\} \tag{7}$$

we obtain [4]

$$E\{\dot{\eta}h\}-\beta_{ech}E\{\dot{\eta}^2\}-\gamma_{ech}E\{\eta\dot{\eta}\}=0, \tag{8}$$

$$E\{\eta h\}-\beta_{ech}E\{\eta\dot{\eta}\}-\gamma_{ech}E\{\eta^2\}=0. \tag{9}$$

The displacement variance [1,2,3] of the system under Gaussian white noise excitation can be expressed as

$$\sigma_\eta^2=\frac{1}{m}\int_{-\infty}^{\infty}\frac{S_0}{\left(\frac{E\{\eta h\}}{E\{\eta^2\}}-\omega^2\right)^2+\omega^2\left(\frac{E\{\dot{\eta}h\}}{E\{\dot{\eta}^2\}}\right)^2}d\omega, \tag{10}$$

The frequency response function [5,6] of the single degree of freedom system is

$$H(\omega)=\frac{1}{m\left(-\omega^2+i\omega\frac{E\{\dot{\eta}h\}}{E\{\dot{\eta}^2\}}+\frac{E\{\eta h\}}{E\{\eta^2\}}\right)}. \tag{11}$$

The power spectral density of the response [1,2,3] is

$$S_\eta(\omega)=\frac{S_F(\omega)}{m^2\left[\left(\frac{E\{\eta h\}}{E\{\eta^2\}}-\omega^2\right)^2+\omega^2\left(\frac{E\{\dot{\eta}h\}}{E\{\dot{\eta}^2\}}\right)^2\right]}. \tag{12}$$

Obtains for the displacement variance

$$\sigma_\eta^2=\pi S_0\frac{E\{\eta^2\}E\{\dot{\eta}^2\}}{mE\{\eta h\}E\{\dot{\eta}h\}}. \tag{13}$$

and the velocity variance is:

$$\sigma_{\dot{\eta}}^2=E\{\dot{\eta}^2\}=R_{\dot{\eta}}(0)=\frac{1}{m}\int_{-\infty}^{\infty}\omega^2\left|H(\omega)\right|^2S_0d\omega=\pi S_0\frac{E\{\dot{\eta}^2\}}{mE\{\dot{\eta}h\}} \tag{14}$$

2. EXAMPLE: THE RANDOM DUFFING OSCILLATOR

For $m=1kg,\ k=25\dfrac{N}{m},\ \phi_1=3{\cdot}10^3\,s^2/m^2,\ c=1,25\dfrac{N\cdot s}{m},\ \phi_1=10,14{\cdot}10^2\,s^4/m^4,\ \varepsilon=0,01$ and the spectral density for excitation $S_F=0,85N^2\cdot s$, obtain for the displacement variance

$$\sigma^2_\eta=\frac{\pi S_0}{m}\frac{1}{2\xi p^3(1+3\alpha\sigma^2_\eta)\left[1+\varepsilon\left(\dfrac{57}{4}\phi_1\sigma^2_\eta+\dfrac{2765}{16}\phi_2\sigma^4_\eta\right)\right]}. \tag{15}$$

The damping characteristic is

$$\beta_{ech}=2\xi_e p_e=\frac{E\{\dot\eta h\}}{E\{\dot\eta^2\}}=2\xi p\left[1+\varepsilon\left(\phi_1\frac{E\{\dot\eta^4\}}{E\{\dot\eta^2\}}+\phi_2\frac{E\{\dot\eta^6\}}{E\{\dot\eta^2\}}\right)\right]+p^2\frac{E\{\dot\eta\eta\}}{E\{\dot\eta^2\}}+\alpha p^2\frac{E\{\dot\eta\eta^3\}}{E\{\dot\eta^2\}}, \tag{16}$$

or:

$$\beta_{ech}=2\xi_e p_e=2\xi p\left[\frac{57}{4}\phi_1\sigma^2_\eta+\frac{2765}{16}\phi_2\sigma^4_\eta\right], \tag{17}$$

because

$$E\{\dot\eta\,\eta\}=0,. \tag{18}$$

$$E\{\dot\eta\,\eta^3\}=0, \tag{19}$$

$$\frac{E\{\dot\eta^6\}}{E\{\dot\eta^4\}}=\frac{63}{4}\sigma^2_\eta, \tag{20}$$

$$\frac{E\{\dot\eta^4\}}{E\{\dot\eta^2\}}=\frac{45}{4}\sigma^2_\eta \tag{21}$$

The elastic characteristic is given by

$$\gamma_{ech}=p_e^2=\frac{E\{\eta h\}}{E\{\eta^2\}}=2\xi p\left[\frac{E\{\eta\dot\eta\}}{E\{\eta^2\}}+\varepsilon\left(\phi_1\frac{E\{\eta\dot\eta^3\}}{E\{\eta^2\}}+\phi_2\frac{E\{\eta\dot\eta^5\}}{E\{\eta^2\}}\right)\right]+p^2\frac{E\{\eta^2\}}{E\{\eta^2\}}+\alpha p^2\frac{E\{\eta^4\}}{E\{\eta^2\}}, \tag{22}$$

or

$$\gamma_{ech}=p_e^{\,2}=p^2(1+3\alpha\sigma^2_\eta), \tag{23}$$

because

$$\frac{E\{\eta^4\}}{E\{\eta^2\}}=3\sigma^2_\eta,\quad . \tag{24}$$

$$\frac{E\{\eta\dot\eta^3\}}{E\{\eta^2\}}\cong 0,\quad \frac{E\{\eta\dot\eta^5\}}{E\{\eta^2\}}\cong 0 \tag{25}$$

The linear equation for the random excitation is

$$\ddot\eta(t)+2\xi p\left[1+\varepsilon\left(\frac{57}{4}\phi_1\sigma^2_\eta+\frac{2765}{16}\phi_2\sigma^4_\eta\right)\right]\dot\eta(t)+p^2(1+3\alpha\sigma^2_\eta)\eta(t)=f(t). \tag{26}$$

The frequency response function [4,5] of the single degree of freedom system is

$$\left|H(\omega)\right|=\frac{1}{m\sqrt{\left(p^2+3\alpha p^2\sigma^2_\eta-\omega^2\right)^2+4\xi^2p^2\omega^2\left[1+\varepsilon\left(\dfrac{57}{4}\phi_1\sigma^2_\eta+\dfrac{2765}{16}\phi_2\sigma^4_\eta\right)\right]^2}} \tag{27}$$

The displacement variance [2,3] of the system under Gaussian white noise excitation can be expressed as

$$\sigma^2_\eta = \pi S_0 \frac{1}{2\xi p^3(1+3\alpha\sigma^2_\eta)\left[1+\varepsilon\left(\frac{57}{4}\phi_1\sigma^2_\eta + \frac{2765}{16}\phi_2\sigma^4_\eta\right)\right]}. \quad (28)$$

Obtain

$$\sigma^2_\eta = 0,018 m^2. \quad (29)$$

The velocity variance is given by

$$\sigma^2_{\dot\eta} = p^2\sigma^2_\eta(1+3\alpha\sigma^2_\eta) = 65\cdot 10^{-2}\frac{m^2}{s^2}. \quad (30)$$

In figure 1. the power spectral density of the response are given for various combinations of the parameter.

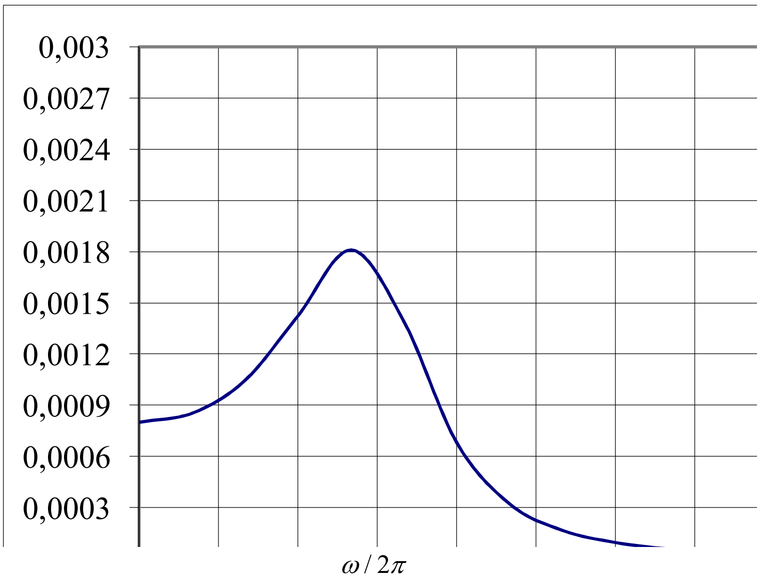

Fig. 1. The power of spectral density $S_\eta[m^2 \cdot s]$ for $m = 1kg, k = 25\frac{N}{m}, c = 1,25\frac{Ns}{m}$.

3. CONCLUSION

Detailed numerical results are presented for of nonlinear oscillators under white noise excitation. Note that the maximum spectral power density values are obtained for velocity.0,8 rad/s. Increases are pronounced in the frequency 0,4…0,8 rad/s and then slowly declines occurring in frequency band 1,6…2,4 rad/s.

Efficient equivalent linear systems with random coefficients for approximating the power spectral density can be deduced. The resonant peak is described very satisfactorily by the approximate solution.

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