



MATHEMATICAL MODEL OF THE DUMMY NECK INCLUDED IN A FRONTAL IMPACT TESTING SYSTEM

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Abstract: Biomechanics is essentially the science that uses mechanical laws in studying biological structures such as muscles, ligaments, joints and other body structures. It is important in understanding the mechanism of injury to the cervical spine (C1-C7). We developed a mathematical approach to mechanical behavior of cervical vertebrae and inter-vertebral discs, in terms of laboratory testing. The paper presents the mathematical model based on Lagrange equations for a frontal impact testing cranio-cervical system. The steps for the solving these equations are exposed in the paper. These studies were combined with FEA and dynamic analysis for determination of constants, forces and functional relations. Results and conclusions are analyzed in the final.

Keywords: mathematical model, dummy, frontal impact, neck testing system, virtual testing system

1. INTRODUCTION

Biomechanics is essentially the science that uses mechanical laws in studying biological structures such as muscles, ligaments, joints and other body structures. It is important in understanding the mechanism of injury to the cervical spine (C1-C7). We developed a mathematical approach to mechanical behavior of cervical vertebrae and intervertebral discs, in terms of laboratory testing.

2. PREPARING THE MATHEMATICAL MODEL

For this study was considered the three-dimensional system consisting of model used to design the test system. The combined dummy neck, base flanges (considered fixed) and two spherical metal caps have mechanical and mathematical model presented in Figure 1. The entire model works with 13 outside forces:

- A system external force F_e which simulates the force acting on the head (impact force) of the form:

$$\overline{F}_e(t) = \begin{cases} (F_{ex} \cdot \frac{t}{t_1}) \cdot \vec{i} + (F_{ez} \cdot \frac{t}{t_1}) \cdot \vec{k}, & \text{pentru } t \in [0; t_1] \\ (F_{ex} \cdot \frac{t_2 - t}{t_2 - t_1}) \cdot \vec{i} + (F_{ez} \cdot \frac{t_2 - t}{t_2 - t_1}) \cdot \vec{k}, & \text{pentru } t \in (t_1; t_2] \\ 0 \cdot \vec{i} + 0 \cdot \vec{j} + 0 \cdot \vec{k}, & \text{pentru } t \in (t_2; t_3] \end{cases} \quad (1)$$

where: F_{ex} - maximum component that acts on OX axis dummy head;

t_1 - the time for which the impact force increases to the maximum;

t_2 - time for which the impact force decreases to the minimum;

t_3 - time period after the system impact.

- Forces of gravity G_1, G_2, G_3, G_4 , of the mechanical system components, like:

$$\begin{aligned} \overline{G}_1(t) &= -(mv_1 + md_1) \cdot g \cdot \vec{k} \\ \overline{G}_2(t) &= -(mv_2 + md_2) \cdot g \cdot \vec{k} \\ \overline{G}_3(t) &= -(mv_3 + md_3) \cdot g \cdot \vec{k} \\ \overline{G}_4(t) &= -(mv_4 + md_4 + mf_4 + ms_4 + msf_4) \cdot g \cdot \vec{k} \end{aligned} \quad (2)$$

where:

mv_1 - the mass of metallic disc 1 ;

md_1 - the mass of elastic disc 1 (rubber or silicone);

mv_2 – the mass of metallic disc 2 ;
 md_2 – the mass of elastic disc 2 (rubber or silicone);
 mv_3 – the mass of metallic disc 3 ;
 md_3 – the mass of elastic disc 3 (rubber or silicone);
 mv_4 – the mass of metallic disc 4 ;
 md_4 – the mass of elastic disc 4 (rubber or silicone);
 ma_4 – the mass of joint item 4 ;
 ms_4 – the mass of lower spherical cap of the element 4;
 msf_4 – the mass of upper spherical cap of the element 4;
 \bar{k} - unit vector of OZ axis.

- Longitudinal forces due to elastic or silicone rubber discs, like:

$$\begin{aligned}
 \overline{F_{e1}}(t) &= k_1 \cdot q_1 \cdot \bar{k} \\
 \overline{F_{e2}}(t) &= k_2 \cdot q_2 \cdot \bar{k} \\
 \overline{F_{e3}}(t) &= k_3 \cdot q_3 \cdot \bar{k} \\
 \overline{F_{e4}}(t) &= k_4 \cdot q_4 \cdot \bar{k}
 \end{aligned}
 \tag{3}$$

where: k_i – longitudinal elastic constants corresponding to elastic forces F_{ei} ;
 q_i – longitudinal generalized coordinates ($i=1\dots4$).

- Transversal elastic forces due to elastic or silicone rubber discs, like:

$$\begin{aligned}
 \overline{F_{1t}}(t) &= k_{1t} \cdot q_{1t} \cdot \bar{i} \\
 \overline{F_{2t}}(t) &= k_{2t} \cdot q_{2t} \cdot \bar{i} \\
 \overline{F_{3t}}(t) &= k_{3t} \cdot q_{3t} \cdot \bar{i} \\
 \overline{F_{4t}}(t) &= k_{4t} \cdot q_{4t} \cdot \bar{i}
 \end{aligned}
 \tag{4}$$

Where: k_{it} – transversal elastic constants corresponding to elastic forces F_{it} ;
 q_{it} – transversal generalized coordinates ($i=1\dots4$).

The mechanical model of the studied assembly was shown in Figure 1. Were chosen as generalized coordinates the coordinates: $q_1, q_2, q_3, q_{1t}, q_{2t}, q_{3t}$. The systems of coordinates $O_1x_1y_1z_1, O_2x_2y_2z_2, O_3x_3y_3z_3$ si $O_4x_4y_4z_4$ are fixed on the elements.

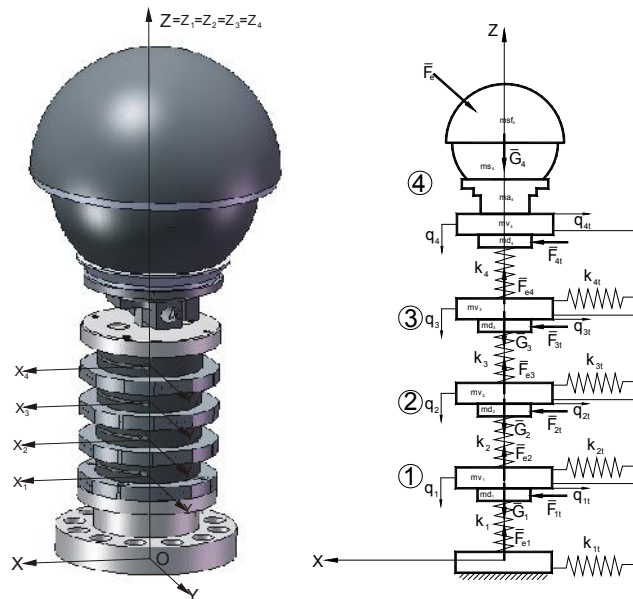


Figure 1: The mechanical model of the studied assembly

In Figure 1 the elastic discs were replaced by rigid discs and by longitudinal and transversal springs and longitudinal constants like k and transversal constants like k_t to be determined by virtual testing.

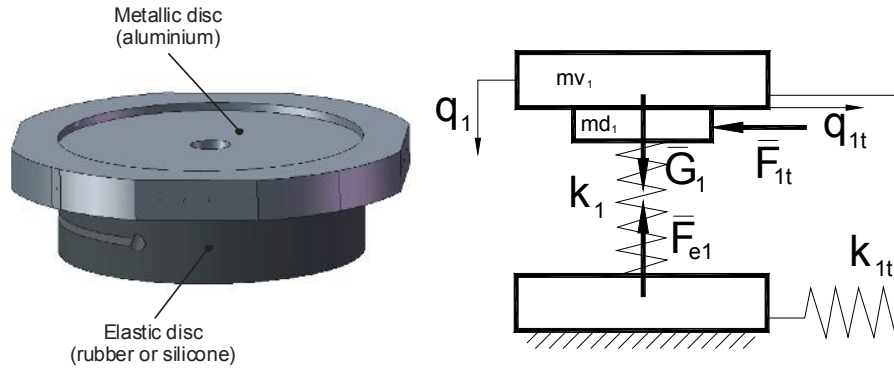


Figure 2: Mechanical equivalent of the elastic discs

To obtain the Lagrange equations is necessary to express the coordinates of the center of mass x_{G1}, y_{G1}, z_{G1} depending of generalized coordinates presented in Figure 2 [1], [2], [3].

Thus, can be written:

$$\begin{bmatrix} x_{G1} \\ y_{G1} \\ z_{G1} \end{bmatrix} = \begin{bmatrix} x_{C1} \\ y_{C1} \\ z_{C1} \end{bmatrix} + \begin{bmatrix} c_{xx1} & c_{xy1} & c_{xz1} \\ c_{yx1} & c_{yy1} & c_{yz1} \\ c_{zx1} & c_{zy1} & c_{zz1} \end{bmatrix} \cdot \begin{bmatrix} x_{m1} \\ y_{m1} \\ z_{m1} \end{bmatrix} \quad (5)$$

where: x_{G1}, y_{G1}, z_{G1} – the coordinates of the mass center for the element 1;

x_{C1}, y_{C1}, z_{C1} – the coordinates of the origin O_1 of the composed element 1 reported to the world system $Oxyz$;

x_{m1}, y_{m1}, z_{m1} – the coordinates of the mass center for the element 1 reported to the own system $O_1x_1y_1z_1$;

$c_{ut} = \cos(u, t)$ – the director cosinus of the u and t axis.

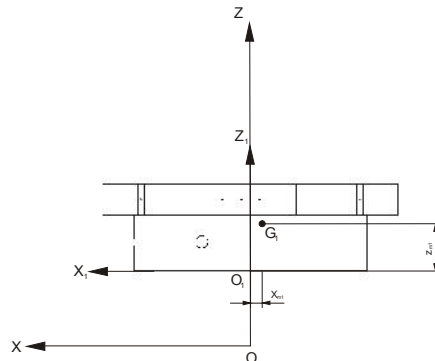


Figure 2: The lateral view of the element 1

Knowing that

$$y_{m1} = 0, y_{C1} = 0, x_{C1} = 0 \quad (6)$$

can be written:

$$\begin{bmatrix} x_{G1} \\ y_{G1} \\ z_{G1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z_{C1} \end{bmatrix} + \begin{bmatrix} c_{xx1} & c_{xy1} & c_{xz1} \\ c_{yx1} & c_{yy1} & c_{yz1} \\ c_{zx1} & c_{zy1} & c_{zz1} \end{bmatrix} \cdot \begin{bmatrix} x_{m1} \\ 0 \\ z_{m1} \end{bmatrix} \quad (7)$$

In this case:

$$\begin{bmatrix} x_{G1} \\ y_{G1} \\ z_{G1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ z_{C1} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x_{m1} \\ 0 \\ z_{m1} \end{bmatrix} \quad (8)$$

That is, it can express the coordinates of center of mass of a compound element 1:

$$\begin{cases} x_{G1} = x_{m1} \\ y_{G1} = 0 \\ z_{G1} = z_{C1} + z_{m1} \end{cases} \quad (9)$$

Taking into account the chosen generalized coordinates q_k equations (9) become:

$$\begin{cases} x_{G1} = -q_1 - x_{m1} \\ y_{G1} = 0 \\ z_{G1} = -q_1 + z_{m1} \end{cases} \quad (10)$$

After the derivation of the equation (10) we obtain:

$$\begin{cases} \dot{x}_{G1} = -\dot{q}_{1r} \\ \dot{y}_{G1} = 0 \\ \dot{z}_{G1} = -\dot{q}_1 \end{cases} \quad (11)$$

Noting:

$$m_1 = mv_1 + md_1 \quad (12)$$

The kinetic energy of the element 1 corresponding to the generalized coordinates is given by:

$$T_1 = \frac{1}{2} m_1 [(\dot{x}_{G1})^2 + (\dot{y}_{G1})^2 + (\dot{z}_{G1})^2] \quad (13)$$

Replacing coordinates derivatives of the mass centers in the kinetic energy equation, we obtain:

$$T_1 = \frac{1}{2} m_1 [(-\dot{q}_{1r})^2 + (-\dot{q}_1)^2] = \frac{1}{2} m_1 [(\dot{q}_{1r})^2 + (\dot{q}_1)^2] \quad (14)$$

For the calculation of generalized forces is expressed, for the start, virtual elementary mechanical work δW_1 :

$$\delta W_1 = \bar{F}_1 \cdot \delta \bar{r}_1 + \bar{M}_1 \cdot \delta \bar{\theta}_1 \quad (15)$$

where: \bar{F}_1 - total external force acting on the element 1;

$\delta \bar{r}_1$ - elementary virtual displacement of the position vector \bar{r}_1 ;

\bar{M}_1 - total external torque moment acting on the element 1;

$\delta \bar{\theta}_1$ - elementary virtual displacement of the angle $\bar{\theta}_1$.

In the case of the studied model, the torque acting on element 1 was considered invalid, so, the virtual elementary mechanical work became:

$$\delta W_1 = \bar{F}_1 \cdot \delta \bar{r}_1 \quad (16)$$

The external force system acting on element 1 is:

$$\bar{F}_1 = -k_{1r} \cdot q_{1r} \cdot \bar{i} + [k_1 \cdot q_1 - (mv_1 + md_1) \cdot g] \cdot \bar{k} \quad (17)$$

The elementary virtual displacement of the position vector \bar{r}_1 is:

$$\delta \bar{r}_1 = \delta x_{G1} \bar{i} + \delta y_{G1} \bar{j} + \delta z_{G1} \bar{k} \quad (18)$$

We can write:

$$\begin{cases} \delta x_{G1} = \delta(-q_{1r} - x_{m1}) = -\delta q_{1r} \\ \delta y_{G1} = 0 \\ \delta z_{G1} = \delta(-q_1 + z_{m1}) = -\delta q_1 \end{cases} \quad (19)$$

And:

$$\delta \bar{r}_1 = -\delta q_{1r} \bar{i} - \delta q_1 \bar{k} \quad (20)$$

In this situation, we can express the virtual elementary mechanical work for element 1:

$$\bar{F}_1 \cdot \delta \bar{r}_1 = \{-k_{1r} \cdot q_{1r} \cdot \bar{i} + [k_1 \cdot q_1 - (mv_1 + md_1) \cdot g] \cdot \bar{k}\} \cdot (-\delta q_{1r} \bar{i} - \delta q_1 \bar{k}) \quad (21)$$

The virtual elementary mechanical work for element 1 became:

$$\delta W_1 = \bar{F}_1 \cdot \delta \bar{r}_1 = k_{1r} \cdot q_{1r} \cdot \delta q_{1r} - (k_1 \cdot q_1 - m_1 \cdot g) \cdot \delta q_1 \quad (22)$$

Similarly, the calculus was made for the elements 2, 3 and 4.

Starting from the equation of the T kinetic energy we calculate partial derivatives like $\frac{\partial T}{\partial q_k}$, $\frac{\partial T}{\partial \dot{q}_k}$ and then $\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right)$.

Knowing that the Lagrange equations has the next shape [1], [2], [3]:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} = Q_k \quad (23)$$

where: T – the kinetic energy of the mechanical system;

q_k – the current generalized coordinate;

Q_k – the current generalized force.

We can express, for the start, the next partial derivatives:

$$\frac{\partial T}{\partial \dot{q}_k} = m_k \dot{q}_k \quad (24)$$

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) = m_k \ddot{q}_k \quad (25)$$

Next, we can express the terms like $\frac{\partial T}{\partial q_k}$:

$$\frac{\partial T}{\partial q_k} = 0 \quad (26)$$

Generalized forces are given by the equation $Q_{q_i} = \frac{\delta W}{\delta q_i}$:

$$\begin{aligned}
 Q_{q_1} &= -k_1 \cdot q_1 + m_1 \cdot g \\
 Q_{q_2} &= -k_2 \cdot q_2 + m_2 \cdot g \\
 Q_{q_3} &= -k_3 \cdot q_3 + m_3 \cdot g \\
 Q_{q_4} &= \begin{cases} (-k_4 \cdot q_4 + m_4 \cdot g) + F_{ex} \cdot \frac{t}{t_1}, & \text{pentru } t \in [0; t_1] \\ (-k_4 \cdot q_4 + m_4 \cdot g) + F_{ex} \cdot \frac{t_2 - t}{t_2 - t_1}, & \text{pentru } t \in (t_1; t_2] \\ -k_4 \cdot q_4 + m_4 \cdot g, & \text{pentru } t \in (t_2; t_3] \end{cases} \\
 Q_{q_{1t}} &= -k_{1t} \cdot q_{1t} \\
 Q_{q_{2t}} &= -k_{2t} \cdot q_{2t} \\
 Q_{q_{3t}} &= -k_{3t} \cdot q_{3t} \\
 Q_{q_{4t}} &= \begin{cases} -k_{4t} \cdot q_{4t} - F_{ex} \cdot \frac{t}{t_1}, & \text{pentru } t \in [0; t_1] \\ -k_{4t} \cdot q_{4t} - F_{ex} \cdot \frac{t_2 - t}{t_2 - t_1}, & \text{pentru } t \in (t_1; t_2] \\ -k_{4t} \cdot q_{4t}, & \text{pentru } t \in (t_2; t_3] \end{cases}
 \end{aligned} \tag{27}$$

Using the previous equations we express the Lagrange equations:

$$\begin{aligned}
 m_1 \ddot{q}_1 &= -k_1 \cdot q_1 + m_1 \cdot g \\
 m_2 \ddot{q}_2 &= -k_2 \cdot q_2 + m_2 \cdot g \\
 m_3 \ddot{q}_3 &= -k_3 \cdot q_3 + m_3 \cdot g \\
 m_4 \ddot{q}_4 &= \begin{cases} (-k_4 \cdot q_4 + m_4 \cdot g) + F_{ex} \cdot \frac{t}{t_1}, & \text{pentru } t \in [0; t_1] \\ (-k_4 \cdot q_4 + m_4 \cdot g) + F_{ex} \cdot \frac{t_2 - t}{t_2 - t_1}, & \text{pentru } t \in (t_1; t_2] \\ -k_4 \cdot q_4 + m_4 \cdot g, & \text{pentru } t \in (t_2; t_3] \end{cases} \\
 m_1 \ddot{q}_{1t} &= -k_{1t} \cdot q_{1t} \\
 m_2 \ddot{q}_{2t} &= -k_{2t} \cdot q_{2t} \\
 m_3 \ddot{q}_{3t} &= -k_{3t} \cdot q_{3t} \\
 m_4 \ddot{q}_{4t} &= \begin{cases} -k_{4t} \cdot q_{4t} - F_{ex} \cdot \frac{t}{t_1}, & \text{pentru } t \in [0; t_1] \\ -k_{4t} \cdot q_{4t} - F_{ex} \cdot \frac{t_2 - t}{t_2 - t_1}, & \text{pentru } t \in (t_1; t_2] \\ -k_{4t} \cdot q_{4t}, & \text{pentru } t \in (t_2; t_3] \end{cases}
 \end{aligned} \tag{28}$$

3. THE RESULTS OF THE SOLVING OF THE LAGRANGE EQUATIONS. RESULTS. CONCLUSIONS

The system of differential equations (28) has the following solutions for intermediate rubber discs (k_i constants were determined using virtual experimental test based on finite element analysis [4], [5]), pendulum mass $m = 30$ kg initial pendulum angle = 30° (pendulum mass and angle determine the force F presented in Figure 1):

$$\begin{aligned}
 q_1(t) &= 5.072 \cdot 10^{-6} \cdot \sin(11.988 \cdot t) \\
 q_2(t) &= 5.072 \cdot 10^{-6} \cdot \sin(11.988 \cdot t) \\
 q_3(t) &= 5.072 \cdot 10^{-6} \cdot \sin(11.988 \cdot t) \\
 q_4(t) &= 1.01355 \cdot 10^{-4} \cdot \sin^2(32.165 \cdot t) \\
 q_{1t}(t) &= -4.5275 \cdot 10^{-6} \cdot \sin(29.985 \cdot t) \\
 q_{2t}(t) &= -4.5275 \cdot 10^{-6} \cdot \sin(29.985 \cdot t) \\
 q_{3t}(t) &= -4.5275 \cdot 10^{-6} \cdot \sin(29.985 \cdot t) \\
 q_{4t}(t) &= \begin{cases} -0.021 \cdot \sin(31.41 \cdot t), & t \in [0, 0.1] \\ -1.24 \cdot 10^{-5} + 2.4484 \cdot 10^{-3} \cdot \sin(270.775 \cdot t - 27.0775), & t \in (0.1, 0.15] \end{cases}
 \end{aligned} \tag{29}$$

Equations (29) define the entire kinematical behaviour of the experimental testing system, but we are very interested by the behaviour of the center of mass of element 4. In the next diagrams were presented the components of the positions, velocities and accelerations on OX and OZ axis for intermediate discs made by rubber.

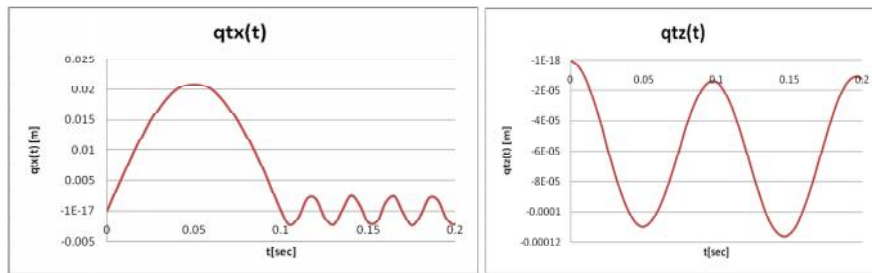


Figure 3: OX and OZ position of the element 4 (head of the dummy)

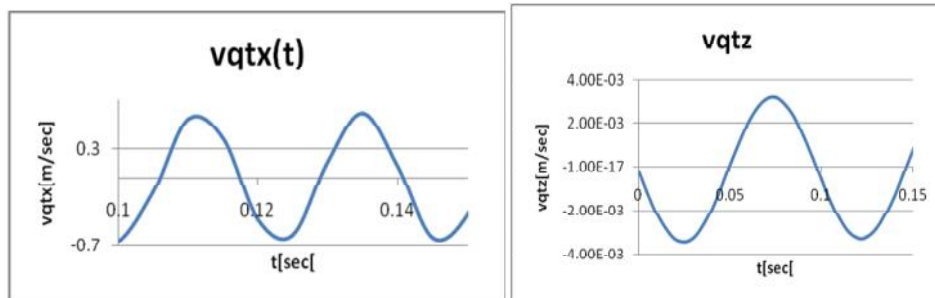


Figure 4: OX and OZ velocities of the element 4 (head of the dummy)

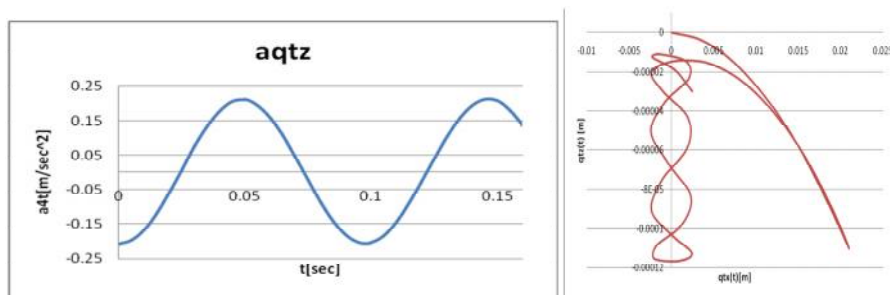


Figure 5: Acceleration and the trajectory of the element 4 (head of the dummy)

Using the same algorithm can be determined the equations for the behavior of the dummy neck using silicone discs. Also, these results will be compared with experimental data obtained on a dummy head-neck testing device.

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