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STUDY OF THE STABILITY FOR A NONLINEAR QUADRATIC SUSPENSION OF A HALF OF AUTOMOBILE

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Abstract: *In this paper is discussed a half of automobile suspension with nonlinear springs for which the elastic force is given by a quadratic function with respect to the elongation. Assuming that both linear elastic and nonlinear elastic elements are compressed one obtains the equilibrium positions for such a suspension and one discusses the stability of these equilibriums. Finally, a numerical application is presented and the diagrams of stability function of the coefficient of the nonlinear force are plotted.*
Keywords: *stability; equilibrium; suspension.*

1. INTRODUCTION

Let be the system in Fig. 1 (the schema of half of an automobile), formed by two equal masses m_1 and a mass m_2. The nonlinear springs (denoted by k_1, ε_1) give an elastic force

$$F_e = k_1 z + \varepsilon_1 z^2 \tag{1}$$

where z is the elongation.

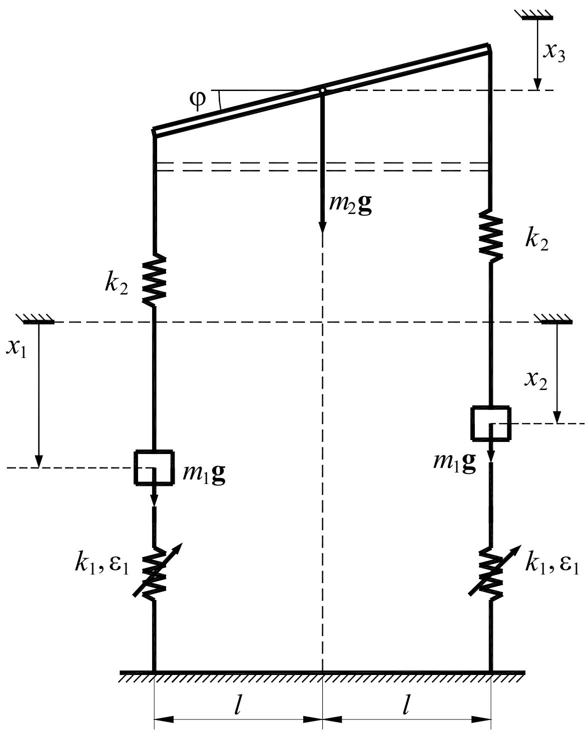


Figure 1: The system.

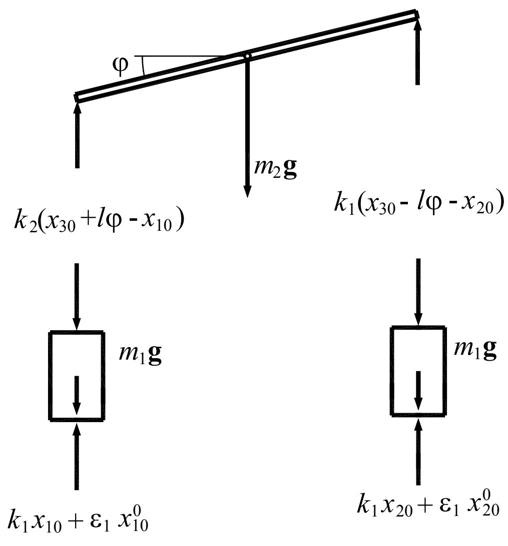


Figure 2: Isolation of the rigid bodies.

The system moves in a vertical plane, the rotation of the bar of mass m_2 (denoted by φ) being considered sufficiently small to may admit the approximations $\sin\varphi \approx \varphi$, $\cos\varphi \approx 1$. Let us suppose that both the nonlinear and the linear springs are contracted. Determine: the positions of equilibrium; their stability as function of the parameter ε_1, assuming that k_1, k_2, m_1, m_2 are known. Numerical application: $m_1 = 50\,\text{kg}$, $m_2 = 750\,\text{kg}$, $l = 2\,\text{m}$, $k_2 = 20000\,\text{Nm}^{-1}$, $k_1 = 10^5\,\text{Nm}^{-1}$, $J = \left[m_2(2l)^2\right]/12 = 1000\,\text{kgm}^2$, $g = 9.8065\,\text{ms}^{-2}$.

2. EQUILIBRIA

The equations of equilibrium read

$$\varepsilon_1 x_{10}^2 + k_1 x_{10} - k_2(x_{30} + l\varphi_0 - x_{10}) - m_1 g = 0\,,\ \varepsilon_1 x_{20}^2 + k_1 x_{20} - k_2(x_{30} - l\varphi_0 - x_{20}) - m_1 g = 0\,,$$
$$k_2(x_{30} + l\varphi_0 - x_{10}) + k_2(x_{30} - l\varphi_0 - x_{20}) - m_2 g = 0\,,\ k_2 l(x_{30} - l\varphi_0 - x_{20}) + k_2 l(x_{30} + l\varphi_0 - x_{10}) = 0\,. \tag{2}$$

where the index 0 corresponds to the position of equilibrium. The above equations may be put in the form

$$\varepsilon_1 x_{10}^2 + (k_1 + k_2)x_{10} - k_2 x_{30} - k_2 l\varphi_0 = m_1 g\,, \tag{3}$$

$$\varepsilon_1 x_{20}^2 + (k_1 + k_2)x_{20} - k_2 x_{30} + k_2 l\varphi_0 = m_1 g\,, \tag{4}$$

$$-k_2 x_{10} - k_2 x_{20} + 2k_2 x_{30} = m_2 g\,, \tag{5}$$

$$x_{10} - x_{20} - 2l\varphi_0 = 0\,. \tag{6}$$

From the relation (6), one obtains

$$\varphi_0 = \frac{x_{10} - x_{20}}{2l}\,, \tag{7}$$

which, replaced in the relations (3) and (4), leads to

$$\varepsilon_1 x_{10}^2 + (k_1 + k_2)x_{10} - k_2 x_{30} - \frac{k_2}{2}(x_{10} - x_{20}) = m_1 g\,, \tag{8}$$

$$\varepsilon_1 x_{20}^2 + (k_1 + k_2)x_{20} - k_2 x_{30} + \frac{k_2}{2}(x_{10} - x_{20}) = m_1 g\,. \tag{9}$$

From (5), we get

$$x_{30} = \frac{m_2 g}{2k_2} + \frac{x_{10} + x_{20}}{2}\,. \tag{10}$$

Subtracting the relations (8) and (9), term by term, it follows that $\varepsilon_1(x_{10}^2 - x_{20}^2) + k_1(x_{10} - x_{20}) = 0$, wherefrom it results

$$x_{10} = x_{20} \text{ or } x_{10} + x_{20} = -\frac{k_1}{\varepsilon_1}\,. \tag{11}$$

If $x_{10} = x_{20}$, then from (7) one obtains $\varphi_0 = 0$, so that from (10) one gets $x_{30} = \frac{m_2 g}{2k_2} + x_{10} = \frac{m_2 g}{2k_2} + x_{20}$. If $x_{10} + x_{20} = -k_1/\varepsilon_1$, then we may write $x_{10} = -\frac{k_1}{\varepsilon_1} - x_{20}$, $x_{20} = -\frac{k_1}{\varepsilon_1} - x_{10}$. The relation (10) leads to $x_{30} = \frac{m_2 g}{2k_2} - \frac{k_1}{2\varepsilon_1}$, while from (7) one obtains $\varphi_0 = \frac{x_{10}}{l} + \frac{k_1}{2l\varepsilon_1}$, $\varphi_0 - \frac{x_{20}}{l} - \frac{k_1}{2l\varepsilon_1}$. The equation (8) takes now the form

$$\varepsilon_1 x_{10}^2 + (k_1 - k_2)x_{10} - \frac{k_1 k_2}{2\varepsilon_1} - \left(m_1 + \frac{m_2}{2}\right)g = 0\,, \tag{12}$$

while the equation (10) becomes

$$\varepsilon_1 x_{20}^2 + (k_1 - k_2)x_{20} - \frac{k_1 k_2}{2\varepsilon_1} - \left(m_1 + \frac{m_2}{2}\right)g = 0\,. \tag{13}$$

As a matter of fact, the equations (12) and (13) are the same. The discriminant of these equations is $\Delta = k_1^2 + k_2^2 + 4\varepsilon_1\left(m_1 + \frac{m_2}{2}\right)g$ and the condition $\Delta \ge 0$ leads to the inequality $\varepsilon_1 \ge -\frac{k_1^2 + k_2^2}{4\left(m_1 + \frac{m_2}{2}\right)g}$. The sum of the roots of the equation (12) (of the equation (13) too) is $S = \frac{k_1 - k_2}{\varepsilon_1} \neq -\frac{k_1}{\varepsilon_1}$, which means that the position of equilibrium (if it exists) is given by

$$x_{10} = x_{20} = \frac{k_2 - k_1 - \sqrt{\Delta}}{2\varepsilon_1} \text{ or } x_{10} = x_{20} = \frac{k_2 - k_1 + \sqrt{\Delta}}{2\varepsilon_1}. \tag{14}$$

As $x_{10} > 0$, $x_{20} > 0$ (the springs are compressed), from $x_{10} + x_{20} = -k_1/\varepsilon_1$ it results $\varepsilon_1 < 0$. It results, from the first equality (14), that $k_2 = \Delta$, wherefrom $k_1^2 + k_2^2 + 4\varepsilon_1\left(m_1 + \frac{m_2}{2}\right)g = k_2^2$, i.e.

$$\varepsilon_1 = -\frac{k_1^2}{4\left(m_1 + \frac{m_2}{2}\right)g}, \tag{15}$$

which verifies the inequalities (xxiii) and $\varepsilon_1 < 0$ too. It results the position of equilibrium

$$x_{10} = x_{20} = -\frac{k_1}{2\varepsilon_1} > 0, \tag{16}$$

ε_1 being given by (15). For the second equation (14), one obtains $k_2 = -\sqrt{\Delta}$, which is absurd. Let us remark that (16) is a particular case of the first relation (11), hence at equilibrium $x_{10} = x_{20}$.

3. EQUATIONS OF MOTION

Using the schema in Fig. 1, these equations read

$$\begin{gathered} m_1\ddot{x}_1 = k_2(x_3 + l\varphi - x_1) - k_1x_1 - \varepsilon_1 x_1^2 + m_1 g,\ m_1\ddot{x}_2 = k_2(x_3 - l\varphi - x_2) - k_1x_2 - \varepsilon_1 x_2^2 + m_1 g, \\ m_2\ddot{x}_3 = k_2(-x_3 - l\varphi + x_1) + k_2(-x_3 + l\varphi + x_2) + m_2 g,\ J\ddot{\varphi} = k_2 l(-x_3 - l\varphi + x_1) - k_2 l(-x_3 + l\varphi + x_2). \end{gathered} \tag{17}$$

Denoting $x_1 = \xi_1$, $x_2 = \xi_2$, $x_3 = \xi_3$, $\varphi = \xi_4$, $\dot{x}_1 = \xi_5$, $\dot{x}_2 = \xi_6$, $\dot{x}_3 = \xi_7$, $\dot{\varphi} = \xi_8$, $a_{10} = -\frac{\varepsilon_1}{m_1}$, $a_{11} = -\frac{k_1 + k_2}{m_1}$, $a_{13} = \frac{k_2}{m_1}$, $a_{14} = \frac{k_2 l}{m_1}$, $a_{20} = -\frac{\varepsilon_1}{m_1}$, $a_{22} = -\frac{k_1 + k_2}{m_1}$, $a_{23} = \frac{k_2}{m_1}$, $a_{24} = -\frac{k_2 l}{m_1}$, $a_{31} = \frac{k_2}{m_2}$, $a_{32} = \frac{k_2}{m_2}$, $a_{33} = -\frac{2k_2}{m_2}$, $a_{41} = \frac{k_2 l}{J}$, $a_{42} = -\frac{k_2 l}{J}$, $a_{44} = -\frac{2k_2 l^2}{J}$,

one obtains the system

$$\begin{gathered} \dot{\xi}_1 = \xi_5,\ \dot{\xi}_2 = \xi_6,\ \dot{\xi}_3 = \xi_7,\ \dot{\xi}_4 = \xi_8,\ \dot{\xi}_5 = a_{10}\xi_1^2 + a_{11}\xi_1 + a_{13}\xi_3 + a_{14}\xi_4 + g, \\ \dot{\xi}_6 = a_{20}\xi_2^2 + a_{22}\xi_2 + a_{23}\xi_3 + a_{24}\xi_4 + g,\ \dot{\xi}_7 = a_{31}\xi_1 + a_{32}\xi_2 + a_{33}\xi_3 + g,\ \dot{\xi}_8 = a_{41}\xi_1 + a_{42}\xi_2 + a_{44}\xi_4. \end{gathered} \tag{18}$$

4. STABILITY OF THE POSITIONS OF EQUILIBRIUM

Denoting by $f_k(\xi_1, \ldots, \xi_8)$, $k = \overline{1,8}$, the expressions of the right member of the relations (18) and by $j_{kl} = \partial f_k / \partial \xi_l$, $k, l = \overline{1,8}$, their partial derivatives, it results the characteristic equation

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 1 \\ j_{51} & 0 & j_{53} & j_{54} & -\lambda & 0 & 0 & 0 \\ 0 & j_{62} & j_{63} & j_{64} & 0 & -\lambda & 0 & 0 \\ j_{71} & j_{72} & j_{73} & 0 & 0 & 0 & -\lambda & 0 \\ j_{81} & j_{82} & 0 & j_{84} & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0, \tag{19}$$

wherefrom

$$\begin{vmatrix} j_{51}-\lambda^2 & 0 & j_{53} & j_{54} \\ 0 & j_{62}-\lambda^2 & j_{63} & j_{64} \\ j_{71} & j_{72} & j_{73}-\lambda^2 & 0 \\ j_{81} & j_{82} & 0 & j_{84}-\lambda^2 \end{vmatrix} = 0 . \tag{20}$$

One obtains the algebraic equation of eighth degree in λ

$$\lambda^8 + A\lambda^6 + B\lambda^4 + C\lambda^2 + D = 0 , \tag{21}$$

where

$$\begin{aligned} &A = -j_{51} - j_{62} - j_{73} - j_{74} , \quad B = j_{62}j_{73} + j_{62}j_{84} + j_{73}j_{84} - j_{64}j_{82} - j_{63}j_{72} \\ &\qquad + j_{51}j_{62} + j_{51}j_{73} + j_{51}j_{84} - j_{53}j_{71} - j_{54}j_{81} , \\ &C = -j_{62}j_{73}j_{84} + j_{64}j_{73}j_{82} + j_{63}j_{72}j_{84} - j_{51}j_{62}j_{73} - j_{51}j_{62}j_{84} - j_{51}j_{73}j_{84} + j_{51}j_{64}j_{82} \\ &\qquad + j_{51}j_{63}j_{72} + j_{53}j_{62}j_{71} + j_{53}j_{71}j_{84} + j_{54}j_{62}j_{81} + j_{54}j_{73}j_{81} , \\ &D = j_{51}j_{62}j_{73}j_{84} - j_{51}j_{64}j_{73}j_{82} - j_{51}j_{63}j_{72}j_{84} + j_{53}j_{64}j_{71}j_{82} - j_{53}j_{64}j_{72}j_{81} \\ &\qquad - j_{53}j_{62}j_{71}j_{84} - j_{54}j_{62}j_{73}j_{81} - j_{54}j_{63}j_{71}j_{82} + j_{54}j_{63}j_{72}j_{81} . \end{aligned} \tag{22}$$

The equation (21), with the notation $u = \lambda^2$, may be written in the form

$$u^4 + Au^3 + Bu^2 + Cu + D = 0 \tag{23}$$

and, for the position of equilibrium be stable, it is necessary and sufficient that all the roots of the equation (23) be negative and distinct (see Discussion). It may occur following situations: the roots are distinct; a root is double; a root is triple; a root is of an order of multiplicity equal to four; two roots are double.

In the case of distinct roots making $u \mapsto -u$ in the equation (24), one obtains $u^4 - Au^3 + Bu^2 - Cu + D = 0$ and, from Descartes's theorem, one deduces the necessary condition of existence of four negative roots $A > 0$, $B > 0$, $C > 0$, $D > 0$. We construct the Sturm sequence associated to the polynomial

$$f(u) = u^4 + Au^3 + Bu^2 + Cu + D . \tag{24}$$

We choose $f_0(u) = u^4 + Au^3 + Bu^2 + Cu + D$, $f_1 = u^3 + \frac{3A}{4}u^2 + \frac{B}{2}u + \frac{C}{4}$. Dividing f_0 by f_1, one obtains the remainder $R_2 = \frac{8B - 3A^2}{16}u^2 + \frac{6C - AB}{8}u + \frac{16D - AC}{16}$. One obtains that is necessary that $8B - 3A^2 \neq 0$; in the opposite case, R_2 would have a degree at most equal to 1 (like the polynomial f_2 in the Sturm sequence) and would result only four terms in the Sturm sequence (f_0, f_1, f_2 and f_3 (the last being a constant)), so that in the sequence $f_0(-\infty)$, $f_1(-\infty)$, $f_2(-\infty)$, $f_3(-\infty)$ we would have at most three variations of sign. It would result that the equation (23) has at most three negative roots, which is not convenient. As a conclusion, it results the necessary condition $8B - 3A^2 \neq 0$. Writing now $R_2 = -\alpha'_2 u^2 - \beta'_2 u - \gamma'_2$, we may choose the following term of Sturm's sequence in the form $f_2(u) = u^2 + \beta_2 u + \gamma_2$, where $\beta_2 = \frac{\beta'_2}{\alpha'_2} = \frac{2(6C - AB)}{8B - 3A^2}$, $\gamma_2 = \frac{\gamma'_2}{\alpha'_2} = \frac{16D - AC}{8B - 3A^2}$.

Dividing now f_1 to f_2, one obtains the remainder $R_3 = -\beta'_3 u - \gamma'_3$, where $\beta'_3 = \gamma_2 - \frac{B}{2} + \beta_2\left(\frac{3}{4}A - \beta_2\right)$, $\gamma'_3 = -\frac{C}{4} + \gamma_2\left(\frac{3}{4}A - \beta_2\right)$. Similar considerations lead to the condition $\beta'_3 \neq 0$, wherefrom $\frac{16D - AC}{8B - 3A^2} - \frac{B}{2} + \frac{2(6C - AB)}{8B - 3A^2}\left[\frac{3}{4}A - \frac{2(6C - AB)}{8B - 3A^2}\right] \neq 0$. We choose $f_3(u) = u + \gamma_3$, with $\gamma_3 = \frac{\gamma'_3}{\beta'_3}$. By dividing f_2 by f_3, it results the remainder $R_4 = \gamma_2 - \gamma_3(\beta_2 - \gamma_3)$ and the polynomial $f_4(u) = \gamma_3(\beta_2 - \gamma_3) - \gamma_2$, which must be nonzero (the roots are distinct!), wherefrom one obtains the condition

$$\frac{-\frac{C}{4} + \gamma_2\left(\frac{3}{4}A - \beta_2\right)}{\gamma_2 - \frac{B}{2} + \beta_2\left(\frac{3}{4}A - \beta_2\right)}\left[\beta_2 - \frac{-\frac{C}{4} + \gamma_2\left(\frac{3}{4}A - \beta_2\right)}{\gamma_2 - \frac{B}{2} + \beta_2\left(\frac{3}{4}A - \beta_2\right)}\right] - \gamma_2 \neq 0 . \tag{25}$$

One may construct following table.

Table 1: Table of the variations of sign in the Sturm sequence.

u	f_0	f_1	f_2	f_3	f_4	W_S
$-\infty$	+	–	+	–	sgn f_4	3 or 4
0	+	+	sgn γ_2	sgn γ_3	sgn f_4	0, 1, 2 or 3

The only possibility to have four negative distinct roots is that $W_S(-\infty)=4$ and $W_S(0)=0$, wherefrom there result the conditions $f_4>0$, $\gamma_2>0$, $\gamma_3>0$.

If the polynomial $f(u)$ given by (24) has a double root, let be $\overline{u}$, then $\overline{u}$ is a root for the derivative $f'(u)$ too, i.e.

$$\overline{u}^4+A\overline{u}^3+B\overline{u}^2+C\overline{u}+D=0\,,\ 4\overline{u}^3+3A\overline{u}^2+2B\overline{u}+C=0\,. \tag{26}$$

The relations (26) multiplied by 4 and $-\overline{u}$, respectively, summed, lead to

$$A\overline{u}^3+2B\overline{u}^2+3C\overline{u}+4D=0\,. \tag{27}$$

We multiply now the second relation (26) by A, the relation (27) by -4 and make the sum, obtaining

$$\left(3A^2-8B\right)\overline{u}^2+\left(2AB-12C\right)\overline{u}+AC-16D=0\,. \tag{28}$$

Summing the relation (27), multiplied by $\left(8B-3A^2\right)$, with the relation (28), multiplied by $A\overline{u}$, one gets the relation

$$\left(4B^2+A^2B-3AC\right)\overline{u}^2+\left(6BC-2A^2C-4AD\right)\overline{u}+D\left(3A^2-8B\right)=0\,. \tag{29}$$

Multiplying the expressions (28) and (29) by $\left(4B^2-A^2B-3AC\right)$ and $\left(8B-3A^2\right)$, respectively, and summing the results thus obtained, one gets

$$\begin{gathered}\left(8AB^3-2A^3B^2-28A^2BC+36AC^2-32ABD-6A^4C+12A^3D\right)\overline{u}\\+4AB^2C-A^3BC-3A^2C^2-128B^2D+64A^2BD+48ACD-9A^4D=0\end{gathered} \tag{30}$$

wherefrom it results $\overline{u}$, with the condition

$$4AB^3-A^3B^2-14A^2BC+18AC^2-16ABD-3A^4C+6A^3D\neq 0\,. \tag{31}$$

We construct now Horner's schema.

Table 2: Horner's schema for a double root.

	1	A	B	C	D
$\overline{u}$	1	$A+\overline{u}$	$\overline{u}^2+A\overline{u}+B$	$\overline{u}^3+A\overline{u}^2+B\overline{u}+C$	0
$\overline{u}$	1	$A+2\overline{u}$	$3\overline{u}^2+2A\overline{u}+B$	0	

The other roots result from $u^2+\left(A+2\overline{u}\right)u+3\overline{u}^2+2A\overline{u}+B=0$ which must have two negative roots, distinct and differing from $\overline{u}$, wherefrom result the conditions $\Delta=\left(A+2\overline{u}\right)^2-4\left(3\overline{u}^2+2A\overline{u}+B\right)>0$, $A+2\overline{u}>0$, $3\overline{u}^2+2A\overline{u}+B>0$, $\dfrac{-\left(A+2\overline{u}\right)\pm\sqrt{\Delta}}{2}\neq\overline{u}$. Writing the relation (31) in the form $E_1\overline{u}+E_2=0$, the notations being obvious, one also obtains the condition $\dfrac{E_2}{E_1}>0$.

In the case of a triple root, let us denote this root by $\overline{\overline{u}}$; then, it must verify the conditions

$$\overline{\overline{u}}^4+A\overline{\overline{u}}^3+B\overline{\overline{u}}^2+C\overline{\overline{u}}+D=0\,,\ 4\overline{\overline{u}}^3+3A\overline{\overline{u}}^2+2B\overline{\overline{u}}+C=0\,,\ 6\overline{\overline{u}}^2+3A\overline{\overline{u}}+B=0\,, \tag{32}$$

Multiplying the second relation (32) by 3, the third one by $-2\overline{\overline{u}}$ and summing, one obtains the equation

$$3A\overline{\overline{u}}^2+4B\overline{\overline{u}}+3C=0\,. \tag{33}$$

Summing now the last relation (32), multiplied by A, to the relation (33), multiplied by -2, it results $\left(3A^2-8B\right)\overline{\overline{u}}+AB-6C=0$, wherefrom $\overline{\overline{u}}=\dfrac{6C-AB}{3A^2-8B}<0$, $3A^2-8B\neq 0$. We construct now Horner's schema.

Table 3: Horner's schema for a triple root.

	1	A	B	C	D
$\overline{\overline{u}}$	1	$A+\overline{\overline{u}}$	$\overline{\overline{u}}^2+A\overline{\overline{u}}+B$	$\overline{\overline{u}}^3+A\overline{\overline{u}}^2+B\overline{\overline{u}}+C$	0
$\overline{\overline{u}}$	1	$A+2\overline{\overline{u}}$	$3\overline{\overline{u}}^2+2A\overline{\overline{u}}+B$	0	

$\overline{\overline{u}}$	1	$A+3\overline{\overline{u}}$	0		

One obtains thus the last root $u^* = -A - 3\overline{\overline{u}} < 0$.

In the case of the root of order of multiplicity equal to four, let be $\overline{\overline{\overline{u}}}$ this root. It will verify the relations $\overline{\overline{\overline{u}}}^4 + A\overline{\overline{\overline{u}}}^3 + B\overline{\overline{\overline{u}}}^2 + C\overline{\overline{\overline{u}}} + D = 0$, $4\overline{\overline{\overline{u}}}^3 + 3A\overline{\overline{\overline{u}}}^2 + 2B\overline{\overline{\overline{u}}} + C = 0$, $6\overline{\overline{\overline{u}}}^2 + 3A\overline{\overline{\overline{u}}} + B = 0$, $4\overline{\overline{\overline{u}}} + A = 0$, wherefrom $\overline{\overline{\overline{u}}} = -\frac{A}{4} < 0$; it results $\left(u + \frac{A}{4}\right)^4 = u^4 + Au^3 + Bu^2 + Cu + D$, wherefrom $B = \frac{3A^2}{8}$, $C = \frac{A^3}{16}$, $D = \frac{A^4}{256}$.

In the case of two double roots, let be $\overline{u} < 0$ and $\overline{\overline{u}} < 0$ the two double roots. One may write $u^4 + Au^3 + Bu^2 + Cu + D = (u - \overline{u})^2 (u - \overline{\overline{u}})^2$, wherefrom $A = -2(\overline{u} + \overline{\overline{u}})$, $B = (\overline{u} + \overline{\overline{u}})^2$, $C = -2\overline{u}\overline{\overline{u}}(\overline{u} + \overline{\overline{u}})$, $D = (\overline{u}\overline{\overline{u}})^2$, $A > 0$, $B > 0$, $C > 0$, $D > 0$. It results that $\overline{u}$ and $\overline{\overline{u}}$ are solutions of the equation $z^2 + \frac{A}{2} z + \sqrt{D} = 0$, i.e. $z_{1,2} = -\frac{A}{4} \pm \frac{1}{2}\sqrt{A^2 - 16\sqrt{D}}$, obtaining thus a new condition $\frac{A^4}{256} > D$. Denoting now $\overline{u} = -\frac{A}{4} + \frac{1}{2}\sqrt{A^2 - 16\sqrt{D}}$, $\overline{\overline{u}} = -\frac{A}{4} - \frac{1}{2}\sqrt{A^2 - 16\sqrt{D}}$, it results $\overline{u} + \overline{\overline{u}} = -\frac{A}{2}$, $\overline{u}\overline{\overline{u}} = \sqrt{D}$, $(\overline{u} + \overline{\overline{u}})^2 = \frac{A^2}{4} = B$, $-2\overline{u}\overline{\overline{u}}(\overline{u} + \overline{\overline{u}}) = A\sqrt{D} = C$.

5. DISCUSSION

Let $\overline{u} = \alpha + \mathrm{i}\beta$, $\alpha \neq 0$, $\beta \neq 0$ be a root of the equation (23), which will be written in the trigonometric form $\overline{u} = |\overline{u}|(\cos\theta + \mathrm{i}\sin\theta)$, wherefrom $\lambda = \overline{u}^{\frac{1}{2}} = \sqrt{|\overline{u}|}\left(\cos\frac{\theta}{2} + \mathrm{i}\sin\frac{\theta}{2}\right)$ or $\lambda = \overline{u}^{\frac{1}{2}} = \sqrt{|\overline{u}|}\left[\cos\left(\frac{\theta}{2} + \pi\right) + \mathrm{i}\sin\left(\frac{\theta}{2} + \pi\right)\right]$. Let us remark that, irrespective of the value of θ, wet get either $\cos(\theta/2) > 0$, or $\cos(\theta/2 + \pi) > 0$, hence the equation (21) will have at least a root with positive real part, i.e. the position of equilibrium is unstable. Let us consider now that a root of the equation (23) is of the form $\overline{u} = \mathrm{i}\beta$, $\beta \neq 0$, i.e. $\overline{u} = |\beta|\left(\cos\frac{\pi}{2} + \mathrm{i}\sin\frac{\pi}{2}\right)$ or $\overline{u} = |\beta|\left(\cos\frac{3\pi}{2} + \mathrm{i}\sin\frac{3\pi}{2}\right)$. We deduce $\lambda = \overline{u}^{\frac{1}{2}} = \sqrt{|\beta|}\left(\cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4}\right)$ or $\lambda = \overline{u}^{\frac{1}{2}} = \sqrt{|\beta|}\left(\cos\frac{3\pi}{4} + \mathrm{i}\sin\frac{3\pi}{4}\right)$ or $\lambda = \overline{u}^{\frac{1}{2}} = \sqrt{|\beta|}\left(\cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4}\right)$, hence at least a root of the characteristic equation (21) has its real part positive, so that the equilibrium is unstable.The case $\alpha = 0$, $\beta = 0$ leads to the root $\overline{u} = 0$, wherefrom it results that $\lambda = 0$ is a double root of the characteristic equation (21). The linear approximation of the motion around the position of equilibrium will contain a term of the form Kt, where K is a constant, hence the equilibrium is unstable too. Thus, the only possibility of stability of equilibrium is that described by the fact that all the roots of the equation (23) are negative. If such a root $\overline{u} < 0$ is double, then for the characteristic equation, one obtains the double roots $\lambda_1 = \mathrm{i}\sqrt{|\overline{u}|}$, $\lambda_2 = -\mathrm{i}\sqrt{|\overline{u}|}$. Each such root leads, in the linear approximation of the motion around the position of equilibrium, to terms of the form $Kt\sqrt{|\overline{u}|}\sin(\pi t/2)$; the equilibrium is unstable too. Hence, it results that the equilibrium is stable (in fact, simply stable) if and only if the four roots of the equation (23) are negative and distinct.

6. NUMERICAL COMPUTATION

One obtains the values $a_{11} = -2400$, $a_{13} = 400$, $a_{14} = 53.333$, $a_{22} = -2400$, $a_{23} = 400$, $a_{24} = -53.333$, $a_{31} = 26.667$, $a_{32} = 26.667$, $a_{33} = -53.333$, $a_{41} = 40$, $a_{42} = -40$, $a_{44} = -160$, $a_{10} = -\frac{\varepsilon_1}{50}$, $a_{20} = -\frac{\varepsilon_1}{50}$, $j_{51} = -\frac{\varepsilon_1\xi_1}{25} - 2400$, $j_{53} = 400$, $j_{54} = 53.333$, $j_{62} = -\frac{\varepsilon_1\xi_2}{25} - 2400 = -\frac{\varepsilon_1\xi_1}{25} - 2400$, $j_{63} = 400$, $j_{64} = -53.333$,

$j_{71} = 26.667$, $j_{72} = 26.667$, $j_{73} = -53.333$, $j_{81} = 40$, $j_{82} = -40$, $j_{84} = -160$. The stability diagrams are plotted in Figs. 3, 4 and 5. One has to consider two branches for $\varepsilon_1 < 0$. The first branch is given by $\xi_{11} = \dfrac{k_2 - k_1 + \sqrt{k_1^2 + k_2^2 + 4\varepsilon_1\left(m_1 + \dfrac{m_2}{2}\right)g}}{2\varepsilon_1}$ and the second one by $\xi_{12} = \dfrac{k_2 - k_1 - \sqrt{k_1^2 + k_2^2 + 4\varepsilon_1\left(m_1 + \dfrac{m_2}{2}\right)g}}{2\varepsilon_1}$.
They exist only if the expression under the radical is positive. The two branches start from the same point for which the expression under the radical vanishes. The first branch may lead, for values of ε_1 sufficiently close to zero, to negative roots ξ_{11}, fact which is not in concordance with the hypothesis that all the springs are compressed. The branch contains simply stable positions of equilibrium and is presented in Fig. 3. The second branch leads to solutions valid for any $\varepsilon_1 < 0$. Moreover, these solutions define simply stable positions of equilibrium. For $\varepsilon_1 \to 0$, one obtains $\xi_{12} \to \infty$. This branch is presented in Fig. 4. If $\varepsilon_1 > 0$, then we have to consider only one branch, described by $\xi_1 = \dfrac{k_2 - k_1 + \sqrt{k_1^2 + k_2^2 + 4\varepsilon_1\left(m_1 + \dfrac{m_2}{2}\right)g}}{2\varepsilon_1}$. This branch leads to $\xi_1 \to \infty$ for $\varepsilon_1 \to 0$ too. It is presented in Fig. 5. If $\varepsilon_1 = 0$, then one obtains the linear case described by $\xi_1 = \dfrac{\left(m_1 + \dfrac{m_2}{2}\right)g}{k_1}$, which is a simply stable position of equilibrium. Obviously, the stability diagram in the general case is much more complicated and to draw it one must take into considerations all the possibilities of compression or expansion of the springs. Moreover, because the function which describes the elastic force in the nonlinear springs is not an odd function, the situations to be studied cannot be obtained one of the other by simply changes of sign. The diagrams which are presented are only parts of the stability diagram of the considered mechanical system.

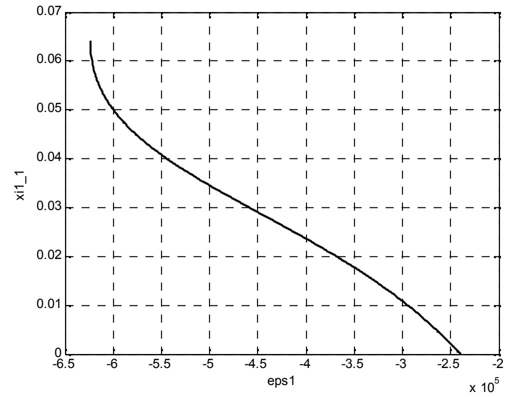


Figure 3: The first branch of stability described by ξ_{11} for $\varepsilon_1 < 0$.

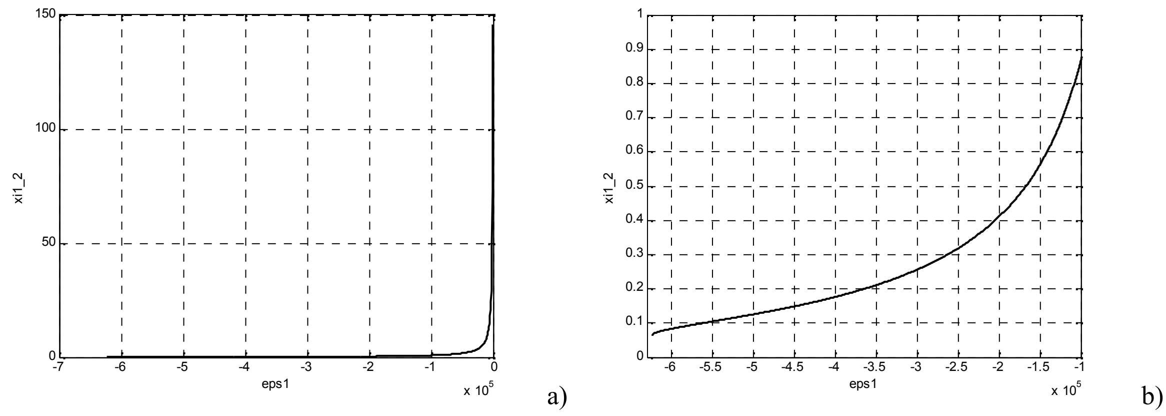
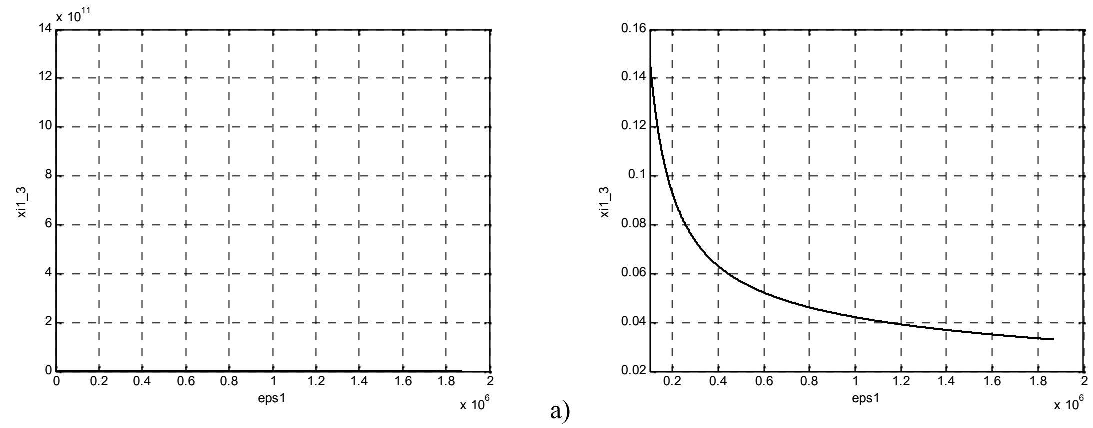


Figure 4: a) The second branch of stability described by ξ_{12} for $\varepsilon_1 < 0$; b) detail of this branch.



a) b)

Figure 5: a) Branch of stability described by ξ_1 for $\varepsilon_1 > 0$; b) detail of this branch.

7. CONCLUSIONS

In this paper we analyzed the dynamics of a half of automobile with quadratic nonlinear springs. We determined the equilibrium positions and we studied the stability of them in the case of all compressed springs. For the particular numerical application we also draw the stability charts.

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