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AN APPROACH TO SOME NON-CLASSICAL EIGENVALUE PROBLEMS OF STRUCTURAL DYNAMICS

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Abstract: *Two main shortcomings of usual formulations, encountered in literature concerning the linear problems of structural dynamics are revealed: the implicit, not discussed, postulation, of the use of Kelvin – Voigt constitutive laws (which is often infirmed by experience) and the calculation difficulties involved by the attempts to use other constitutive laws. In order to surpass these two categories of shortcomings, the use of the bilateral Laplace – Carson transformation is adopted. Instead of the dependence on time, t, of a certain function f (t), the dependence of its image $f^{\#}$ (p) on the complex parameter $p = \chi + i\omega$ (ω: circular frequency) will occur. This leads to the formulation of associated non-classical eigenvalue problems. The basic relations satisfied by the eigenvalues $\lambda_r^{\#}(p)$ and the eigenvectors $\mathbf{v}_r^{\#}(p)$ of dynamic systems are examined (among other, the property of orthogonality of eigenvectors is replaced by the property of pseudo-orthogonality). The case of points p = p', where multiple eigenvalues occur and where, as a rule, chains of principal vectors are to be considered, is discussed. An illustrative case, concerning a non-classical eigenvalue problem, is presented. Plots of variation along the ω axis, for the real and imaginary components of eigenvalues and eigenvectors, are presented. A brief final discussion closes the paper.*
Keywords: ***N****on-classical eigenvalue problems, Laplace – Carson transformation, pseudo-orthogonality, multiple eigenvalues, singular eigenvectors.*

1. INTRODUCTION

The main object of the paper is represented by non-classical eigenvalue problems encountered in the linear dynamics of structures (having, formally, a finite number of degrees of freedom). Matrix formulations are used (vectors: lower case, bold, matrices: upper case bold, characters).
The main tool for calculations, used in the paper, is represented by thebilateral Laplace – Carson transform [9], according to which the relations between the *original functions* $g(t)$ and their corresponding *image functions* $g^{\#}(p)$ are

$$g^{*}(p) = p \int_{-\infty}^{\infty} e^{-pt} g(t)\, dt \qquad \text{where } \chi = \mathrm{Re}\, p \in (\alpha, \beta) \qquad (1.1a)$$

$$g(t) = (1 / 2\pi i) \int_{c-i\infty}^{c-+i\infty} [e^{pt} g^{*}(p) / p]\, dp \qquad \text{where } c \in (\alpha, \beta) \qquad (1.1b)$$

The starting point of the developments of the paper is represented by a view at the linear equation of motion for the original functions,

$$\mathbf{M}\, d^2\mathbf{u} / d\mathbf{t}^2 + \mathbf{C}\, d\mathbf{u} / dt + \mathbf{K}\mathbf{u} = \mathbf{f}(t) \qquad (1.2a)$$

which becomes for the image functions, determined by the bilateral Laplace – Carson transform,

$$(p^2\mathbf{M} + p\,\mathbf{C} + \mathbf{K})\, \mathbf{u}^{*}(p) = \mathbf{f}^{*}(p) \qquad (1.2b)$$

The option for a critical reconsideration of the equation (1.2a) is determined by the fact that the constitutive laws implicitly postulated in the formulation of this equation, which are of Kelvin – Voigt type, lead often to results that are not confirmed by physical experience, while an attempt to adopt a different type of constitutive laws leads to considerable difficulties for the calculation techniques in case one tries to deal with the original functions.

A (rather simplistic) frequently encountered approach in structural dynamics is represented by postulating for the beginning the existence of ideally elastic constitutive law, which lead for the equations (1.2) to a classical eigenvalue problem,

$$(-\mu^2\boldsymbol{M} + \boldsymbol{K})\,\boldsymbol{v} = 0 \qquad (1.3)$$

for which the solution is represented by a system of constant and real eigenvalues $\mu_r$and eigenvectors$\boldsymbol{v}_r$. Thereafter, a correction is introduced just for the eigenvalues, in a way that is similar to the oneusable in case of single degree of freedom systems. This approach keeps the eigenvectors real, while the eigenvalues become complex functions of the variable *p*.

A more correct procedure would lead to the need to consider together all three matrices occurring in the equations (1.2). It is shown[5] that the eigenvalue problem is in general irreducible to an eigenvalue problem for the equation

$$(-\mu^2\boldsymbol{M} + \mathrm{i}\mu\,\boldsymbol{C} + \boldsymbol{K})\,\boldsymbol{v} = 0 \qquad (1.3')$$

It is reducible to that type of equation only in case the matrix $\boldsymbol{C}$ can be represented as a linear combination of terms $[\boldsymbol{K}\,(\boldsymbol{M}^{(-1)}\,\boldsymbol{K})^{\,j}]$, where the values of *j* are integer, arbitrary. In case the eigenvalue problem corresponding to the equation (1.3') is no longer reducible to a classical problem, a correct way, dealt with in literature, is as follows: the non-linear *n*-dimensional problem corresponding to the equation (1.3'), is replaced by a linear, 2*n* – dimensional one. In this latter case the matrices used become usually non-symmetrical, while the solutions become complex. Note that this latter approach is usable only in case Kelvin – Voigt constitutive laws are admitted.

A different approach [7], [8], is adopted in the paper. An image equation

$$[p^2\boldsymbol{M} + \boldsymbol{K}^{\#}(p)]\,\boldsymbol{u}^{*}(p) = \boldsymbol{f}^{*}(p) \qquad (1.4)$$

where, the case a Kelvin – Voigt constitutive law is admitted, is referred to. The matrix $\boldsymbol{K}^{\#}(p)$ would become in this case equal to the matrix $p\,\boldsymbol{C} + \boldsymbol{K}$ of equation (1.2b), dealt with previously. In order to set up the constitutive laws of structural components, the use of generalized Maxwell laws [6] is proposed [7], [8]. The generalized Maxwell law is as follows: an ideally elastic (Hooke) component is connected in parallel with several Maxwell type components. The solution adopted in this way benefits from two main advantages: on one hand, experimentally determined characteristics can be approximated upon a desired interval of the ω-axis; on the other hand, there exist methodological advantages raised by the analytical properties of the laws postulated, characterized by the existence of poles (the matrix$\boldsymbol{K}^{\#}(p)$ is meromorphic)

2. ANALYTICAL DEVELOPMENTS

2.1. Properties of some constitutive laws

To start, adiscussion on some alternative constitutivelaws is dealt with. Theselaws should allowdefining the most appropriate kind of equations of motion for the dynamic systems investigated. Keeping in view the fact that the main tool for analysis is represented bythe use of the bilateral Laplace – Carson transform (1.1), thisapproach is based on the use of the transforms of original functions depending on time.Two basic entities are considered, for illustration of the use of the transforms referred to: the normal stress, $\sigma^{\#}(p)$ and the homologous local deformation $\varepsilon^{\#}(p)$. An explicite extension to tensorial functions is not necessary in this respect.

Two reference models, in which the elasticity (or elastic stiffness) modulus, *E*, and the viscous stiffness modulus, η, intervene, are used:

- the ideally elastic model (called Hooke's model), $\sigma^{\#}(p) = E\,\varepsilon^{\#}(p)$ (2.1a)
- the ideally viscous model (called Newton's model), $\sigma^{\#}(p) = p\,\eta\,\varepsilon^{\#}(p)$ (2.1b)

The parameter *E* is the elasticity modulus, or the modulus of elastic stiffness. The parameter η is the viscous stiffness modulus. These models are to be dealt with in adequate ways for performing specific analyses. The parameters *E* and η are first used for connections in parallel or in series respectively and are to be combined in an appropriate way to correspond to various goals of analysis.

The model of solid body with retardation (called the*Kelvin – Voigt* model):

$$\sigma^{\#}(p) = E\,\varepsilon^{\#}(p) + p\,\eta_{ret}\,\varepsilon^{\#}(p) = (E + p_{ret}\,\eta)\,\varepsilon^{\#}(p) = E\,(1 + p\,\eta_{ret}/E)\,\varepsilon^{\#}(p) = E\,(1 + p\,\tau_{\mathrm{ret}})\,\varepsilon^{\#}(p) \qquad (2.2a)$$

The model of viscous fluid bodywith relaxation (called the *Maxwell* model):

$$\varepsilon^{\#}(p) = \sigma^{\#}(p)/E + p\sigma^{\#}(p)/\eta_{rel}\,,\;\sigma^{\#}(p) = [p\,E\,\eta_{rel}/(E + p\,\eta_{rel})]\varepsilon^{\#}(p) = [E/(1 + p\,\tau_{\mathrm{rel}})]\,\varepsilon^{\#}(p) \qquad (2.2b)$$

The two lattermodels include two parameters of physical dimension time:the retardation time, τ_{ret},and the relaxation time, τ_{rel}, respectively.
A first combination (in parallel) of these two models is the *Poynting* model

$$\sigma^{\#}(p) = [E_0 + E_1 / (1 + p\,\tau_{ret})]\,\varepsilon^{\#}(p) \qquad (2.2c)$$

while a generalization of it is the *generalized Maxwell* model,

$$\sigma^{\#}(p) = [E_0 + \Sigma_k E_k / (1 + p\,\tau_{ret.k})]\,\varepsilon^{\#}(p) \qquad (2.2d)$$

The scalar models (2.1) and (2.2) may be extended to pluri – dimensional models, in order to derive specific laws to structural models. The use of the model corresponding to relation (2.2d) is subsequently preferred. This is because this option makes it possible to approximate the rheological properties of materials and, at the same time, offer controllable singularities (poles of the theory of functions of a complex variable).

NOTE:

1. The fact that the denominators of the terms of expressions (2.2d), as well as of the denominator of expression (2.2c), have real, positive, values leads to the fact that the poles intervening in the expressions of terms of index k are placed on the negative half-axis, Re $p < 0$, Im $p = 0$, at abscissae of $(-1 / \tau_{ret.k}.)$.
2. The coefficients E_k of expression (2.2d) have the same physical dimension as that of the coefficient E_0, while their physical sense is to be specified for each of them.

2.2. Relations of structural dynamics for structures with various constitutive laws

Returning to the equations (2.2), a system of real eigenvectors to simultaneously diagonalize the matrix triad($\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$) of the equation of motion does not exist in the general case for systems having Kelvin – Voigt constitutive laws. The consequence of this fact is that, in the general case when the eigenvectors become complex, a transfer of energy between the free vibrations corresponding to different eigenmodesoccurs. More generally, for a pair of matrices [$\boldsymbol{M}$, $\boldsymbol{K}^{\#}(p)$], where the second matrix is variable, there does not exist a system of constant, real eigenvectors in case the analysis is performed for bilateral Laplace – Carson transforms (where p is the complex, independent, variable, replacing the time argument t,specific to analysis in the field of original functions).
This situation has important consequences, considered subsequently. On one hand, the property of orthogonality of eigenvectors corresponding to different eigenmodes is no longer satisfied. It is consequently appropriate to introduce some new concepts, namely the concepts of *pseudo-orthogonality* and of *pseudo-normalization,* which generalize the classical concepts of orthogonality and normalization.
Given the advantages of use of the solutions of eigenvalue problems derived for the equation of motion, which are illustrated inliterature for various cases, a generalization to the non-classical case was looked for. Two orientations may be distinguished in these studies:
a) approaches which are aiming at the direct determination of singularities (more precisely, the zeroes of the determinant for the image impedance matrix $\boldsymbol{Z}^{\#\wedge}(p)$ [1], [5]);

$$\boldsymbol{Z}^{\#\wedge}(p) = p^2\boldsymbol{M} + \boldsymbol{K}^{\#\wedge}(p) \qquad (2.3)$$

these approaches are usable, practically, in case of adoption of Kelvin – Voigt constitutive laws;
b) approaches aimed at deriving the inverse matrix $\boldsymbol{Z}^{(-1)\#}(p)$ of the matrix $\boldsymbol{Z}^{\#}(p)$, whichis a function of the p variable [7]; this way is the only one dealt with in this paper, due to its more general usability.
Following developments are starting from the equation (1.4), where both matrices $\boldsymbol{M}$ and $\boldsymbol{K}^{\#}(p)$ are symmetrical and lead to the eigenvalues$\lambda_r^{\#}(p)$ depending on the p parameter, for the homogeneous equation

$$[-\lambda^{\#}(p)\,\boldsymbol{M} + \boldsymbol{K}^{\#}(p)]\boldsymbol{v}^{*}(p) = 0 \qquad (2.4)$$

for which the non-trivial solutions (in case the value of variable p to which the solution refers is not affected by singularities), is represented by the eigenvalues $\lambda_r^{\#}(p)$ and the corresponding eigenvectors $\boldsymbol{v}_r^{*}(p)$. The existence of non-trivial solutions implies for the equation zeroes for the determinant of [- $\lambda^{\#}(p)\,\boldsymbol{M} + \boldsymbol{K}^{\#}(p)$],

$$\text{Det}\,\{-\,a^{\#}\boldsymbol{M} + \boldsymbol{K}^{\#}(p)\} = 0 \qquad (2.5)$$

2.3. The case of points $p$where the eigenvalues are different (i.e. simple)

Due to the symmetry properties of matrices, in case of two different eigenvalues $\lambda_r^{\#}(p)$ and $\lambda_s^{\#}(p)$, the corresponding eigenvectors $\boldsymbol{v}_r^{\#}(p)$ and $\boldsymbol{v}_s^{\#}(p)$ are pseudo-orthogonal with respect to both matrices,

$$\boldsymbol{v}_r^{\#}(p)^{\mathrm{T}}\boldsymbol{M}\,\boldsymbol{v}_s^{\#}(p) = 0 \qquad (r \neq s) \qquad (2.6a)$$

$\boldsymbol{v}_r^{\#}(p)^{\mathrm{T}}\boldsymbol{K}^{\#}(p)\, \boldsymbol{v}_s^{\#}(p) = 0 \qquad (r \neq s) \qquad (2.6b)$

(note: orthogonality would have involved that one of the factor vectors should be replaced by its complex conjugate).
In a similar way, the *pseudo-norm* (with respect to matrix $\boldsymbol{M}$) of a vector $\boldsymbol{v}^{\#}(p)$, $m^{\#}(\boldsymbol{v})$, is defined by the relation

$m^{\#}(\boldsymbol{v})^2 = \boldsymbol{v}^{\#}(p)^{\mathrm{T}}\boldsymbol{M}\, \boldsymbol{v}^{\#}(p) \qquad (2.7)$

while the *pseudo-normalized* with respect to matrix $\boldsymbol{M}$homologous vector $\boldsymbol{v}^{)}(p)$, $\boldsymbol{v}^{\#(M)}(p)$, is given by the relation

$\boldsymbol{v}^{\#(M)}(p) = \boldsymbol{v}^{\#}(p) \,/\, m^{\#}(\boldsymbol{v}) \qquad (2.8)$

In order to formulate some condensed relations, it is appropriate to define the matrix of eigenvectors,$\boldsymbol{V}^{\#}(p)$. Its columns are represented by the eigenvectors $\boldsymbol{v}_r^{\#}(p)$ (arranged in the order of eigenvalues of rank (r)). In the same way, it is possible to define a matrix of pseudo-normal eigenvectors, $\boldsymbol{V}^{\#(M)}(p)$. The condition of pseudo-normalization may be rewritten as

$\boldsymbol{v}_r^{\#\,(M)T}(p)\, \boldsymbol{M}\, \boldsymbol{v}_r^{\#\,(M)}(p) = \delta_{\mathrm{rs}} \qquad (\delta_{\mathrm{rs}}$: Kronecker's symbol)) $\qquad (2.9)$

while a homologous relation for the matrix $\boldsymbol{K}^{\#}(p)$ is

$\boldsymbol{v}_r^{\#\,(M)T}(p)\, \boldsymbol{K}^{\#}(p)\, \boldsymbol{v}_s^{\#\,(M)}(p) = [\lambda_r^{\#}(p)\, s_r^{\#}(p)]^{1/2}\delta_{\mathrm{rs}} \qquad (2.10)$

The vectors $\boldsymbol{v}_r^{\#(M)}(p)$ span mono-dimensional subspaces which are invariant with respect to the pair of matrices ($\boldsymbol{M}$, $\boldsymbol{K}^{\#}(p)$).
The relations

$\boldsymbol{V}^{(M)\#T}(p)\, \boldsymbol{M}\, \boldsymbol{V}^{(M)\#}(p) = \boldsymbol{I}_n \qquad$ ($\boldsymbol{I}_n$: unit matrixof dimension n) $\qquad (2.11a)$

$\boldsymbol{V}^{(M)\#T}(p)\, \boldsymbol{K}^{\#}(p)\, \boldsymbol{V}^{(M)\#}(p) = \mathbf{Diag}\ \{\text{ª}_r^{\#}(p)\} \qquad (2.11b)$

$\boldsymbol{V}^{(M)\#T}(p)\, \boldsymbol{M}\, \boldsymbol{K}^{(-1)\#}(p)\, \boldsymbol{M}\, \boldsymbol{V}^{(M)\#}(p) = \mathbf{Diag}\ \{1 \,/\, \text{ª}_r^{\#}(p)\} \qquad (2.11c)$

$\boldsymbol{V}^{(M)\#T}(p)\, \boldsymbol{Z}^{\#}(p)\, \boldsymbol{V}^{(M)\#}(p) = \mathbf{Diag}\ \{p^2 + \text{ª}_r^{\#}(p)\} \qquad (2.11d)$

$\boldsymbol{V}^{(M)\#T}(p)\, \boldsymbol{M}\, \boldsymbol{Z}^{(-1)\#}(p)\, \boldsymbol{M}\, \boldsymbol{V}^{(M)\#}(p) = \mathbf{Diag}\ \{1 \,/\, [p^2 + \text{ª}_r^{\#}(p)]\} \qquad (2.11e)$

(where the symbol**Diag** means a diagonal matrix of dimension n,while the impedance matrix, $\boldsymbol{Z}^{\#}(p)$, is defined by the relation (2.3).
Conversely, the relations (having the sense of spectral expansions for the matrices of the dynamic system dealt with) are

$\boldsymbol{K}^{\#}(p) = \boldsymbol{M}\, \boldsymbol{V}^{(M)\#}(p)\, \mathbf{Diag}\ \{\text{ª}_r^{\#}(p)\}\, \boldsymbol{V}^{(M)\#T}(p)\, \boldsymbol{M} \qquad (2.12a)$

$\boldsymbol{K}^{\#(-1)}(p) = \boldsymbol{V}^{(M)\#}(p)\, \mathbf{Diag}\ \{1 \,/\, \text{ª}_r^{\#}(p)\}\, \boldsymbol{V}^{(M)\#T}(p) \qquad (2.12b)$

$\boldsymbol{Z}^{\#}(p) = \boldsymbol{M}\, \boldsymbol{V}^{(M)\#}(p)\, \mathbf{Diag}\ \{\, p^2 + \text{ª}_r^{\#}(p)\}\, \boldsymbol{V}^{(M)\#T}(p)\, \boldsymbol{M} \qquad (2.12c)$

$\boldsymbol{Z}^{\#(-1)}(p) = \boldsymbol{V}^{(M)\#}(p)\, \mathbf{Diag}\ \{1 \,/\, [p^2 + \text{ª}_r^{\#}(p)]\}\, \boldsymbol{V}^{(M)\#T}(p) \qquad (2.12d)$

It may be shown that the properties accepted for the matrix $\boldsymbol{K}^{\#}(p)$lead to the conclusion that the real and imaginary parts of the eigenvalue $\text{ª}^{\#}_r(p)$ satisfy the conditions

Re $\text{ª}^{\#}_r(p) > 0$ for Re $p \geq 0$ and

Im $\text{ª}^{\#}_r(p)$ / Im $p \geq 0$ for the whole plane $\{p\}$,

while the poles of the eigenvalues $\text{ª}^{\#}_r(p)$can be placed only along the half-axis (Im $p = 0$, Re $p < 0$) in case the scalar constitutive laws of types (2.2) can be directly applied as constitutive laws between the specific vectors of internalforces and the specific deformation components. It may be shown also that the eigenvalues $\text{ª}^{\#}_r(p)$ are stationary at the point $\boldsymbol{v} = \boldsymbol{v}^{\#}_r(p)$, in case one considers the kind of variation of the expression $\text{ª}^{\#}_r(p) = \boldsymbol{v}^T(p)\, \boldsymbol{K}^{\#}(p)\, \boldsymbol{v}^{\#}(p)$ along the *hyper-pseudosphere* $\boldsymbol{v}^T\boldsymbol{M}\boldsymbol{v} = 1$.
The matrix $\boldsymbol{K}^{\#}(p)$ and the eigenvalues $\text{ª}^{\#}_r(p)$ may be expanded into integer series of powers of $(p - p_0)$ [4]. The eigenvectors $\boldsymbol{v}^{\#}_r(p)$ are uniform functions, which may be expanded in a similar way into series of powers. Due to the condition (2.3), and to the fact that a complete basis exists, the eigenvectors do not have, at such points p, zeroes or poles.

2.4. The case of points *p = p'*, where multiple eigenvalues exist

The case of points $p = p'$, where multiple eigenvalues ª$^{\#}_{r}(p)$ exist, raises special problems, which imposea special kind of analysis, needing a revision of the calculation techniques usually adopted for points p, where all eigenvalues are different. Some features of the variation of the eigenvectors in the neighborhood of points $p = p'$ where a multiple eigenvalue ª$^{\#}_{r'}$ (p) exists, may be mentioned: the variation of the system of eigenvectors in the subspace spanned by the system of eigenvectors corresponding to a multiple eigenvalue ª$^{\#}_{r'}$ (p) must be replaced by a chain of principal vectors [10]. The diagonal submatrix corresponding to the chain referred to will no longer be a diagonal one, but is to be replaced by a submatrix of Jordan's canonic type [10]. The relations developed previously in subsection 2.3 of the paper must be correspondingly adapted. As mentioned in the noteof subsection 2.1, item 1, in the problems dealt with in the paperthis is not expected to happen.

3. ILLUSTRATIVE APPLICATION

The physical problem dealt with is related to the examination of the vibration of a dynamic system ***S*** consisting of a mass connected by means of a perfectly elasticspring to a (vertical) axi-symmetrical foundation block that is connected at its turn to the elastic half-space.Note here that the connection to the half-space involves dissipative propertieseven in case of an ideally elastic half-space. This is due to the fact that during the vibration process the energy is radiated from the foundation block, without returning to the contact zone. In order to keep calculations as simple as possible, the dissipative properties of the contact zone are assumed to correspond to a Kelvin – Voigt constitutive model. In agreement with themodelling and the approximate relations given in [2], the contact of an axi-symmetrical block with the half-space is equivalent to a dynamic single degree of freedom system, for which following input data were adopted:

- the mass ofthe rigid foundation block, including the equivalent mass pertaining to the half space material: m_1 = 15 t;
- the viscous stiffness of the system of contact with the half-space: c_1 = 4500 t/s = 4500 kNs/m;
- the elastic stiffness of the same contact system: k_1= 3 000 000 t/s^2 = 3 000 000 kN/m;
- the mass of the upper body: 5 t;
- the viscous stiffness of the contact system between the two bodies:$c_2 = 0$;
- the elastic stiffness of the same: k_2 = 100 000 t / s^2 = 100 000 kN/m

The condition of zero value of the determinant corresponding to the equation of motion is

$$m_1 m_2 \text{ª}^{\#2} - [\, m_1\ k_2 + m_2\ (\, cp + k_1 + k_2)]\ \text{ª}^{\#} + k_1 k_2 = 0 \qquad (3.1)$$

with the solutions for the eigenvalues

$$\text{ª}_{I,II}{}^{\#}(p) = \ < [m_1 k_2 + m_2 (cp + k_1 + k_2)] \ \text{-/+}\ \{[m_1 k_2 + m_2 (cp + k_1 + k_2)]^2 - 4\, m_1 m_2 k_1 k_2\}^{1/2} > / \ (2\, m_1 m_2);$$

and for the eigenvectors respectively

$$v_{1r}{}^{\#} = (k_2 - \text{ª}_r{}^{\#} m_2) \,/\, n_r{}^{\#} \qquad (3.3a)$$

$$v_{2r}{}^{\#} = k_2 \,/\, n_r{}^{\#} \qquad (3.3b)$$

where the denominator n_r# has the expression

$$n_r{}^{\#} = [m_1 (k_2 - \text{ª}_r{}^{\#} m_2)^2 + m_2 k_2{}^2]^{1/2} \qquad (3.4)$$

The solutions (eigenvalues and eigenvectors) as functions of the non-dimensional parameter cµ/ k_1,assuming p = i µ, are presented in Figures 3.1 and 3.2 respectivelyfor the non-dimensional interval (0., 2.0) of cµ/ k_1.
To note that the colors blue (for the real parts) and red (for the imaginary parts) respectively were used.It may be remarkedthat a singularity occurs for the eigenvector $\boldsymbol{v}_{I,}{}^{\#}$at a value of about 0.15 of the non-dimensional ratio cµ/ k_1.Examining the plots presented, it turns out thatfor the system dealt with a strong dependence on the non-dimensional argument exists. Of course, one must take into account the fact that the results presented concern directly the Laplace – Carson images and that a use of them for practical purposes involves in principle a conversion to the field of originals for the functions of interest.

4. FINAL CONSIDERATIONS

The paper presented is dealing with a problem of wide interest, namely that of contributing to the adoption of an instrument of analysis of the performance of dynamic systems having components characterized by linear constitutive laws of a quite high complexity. This may lead to analyses to be more realistic than the practically exclusive use of Kelvin – Voigt constitutive laws, whichare so frequently encountered in the literature, without the required comments.

The use of the bilateral Laplace – Carson transform represents a highly efficient tool of analysis. Learning this procedure is recommended to those engaged in the linear analyses of various problems of structural dynamics.

Besides the direct transform, expressed by the relation (1.1a), which consists of usual integration, the use of the inverse transform (1.1b) based on the residue theorem of the theory of complex functions is highly recommendable.

The appropriate consideration of the convergence band (α, β) that is specific to every functions dealt with, should be carefully carried out.

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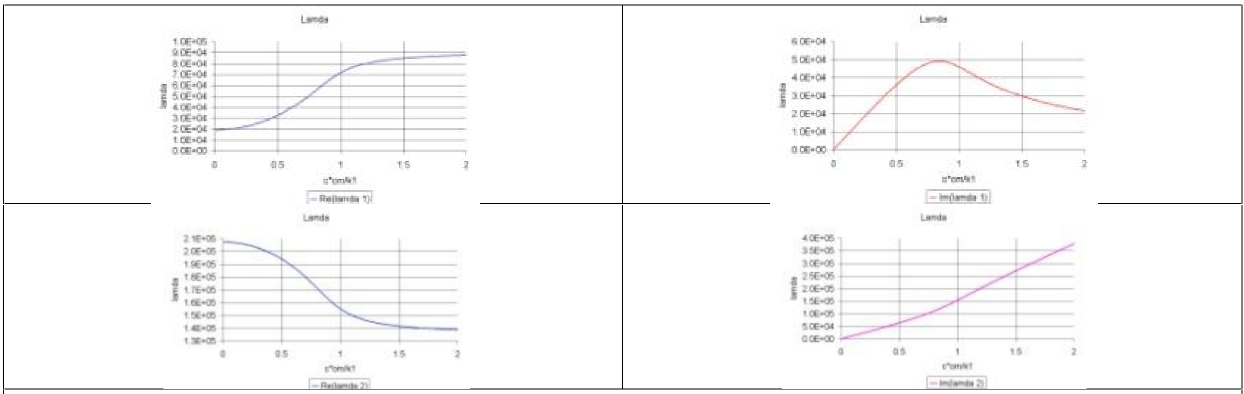

Figure 3.1. Real and imaginary parts of eigenvalues $a_r^{\#}(p)$

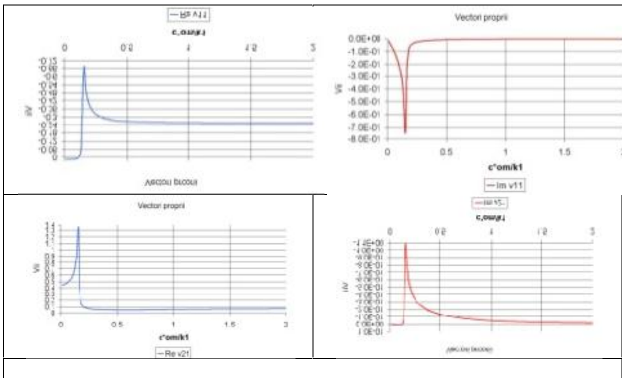

Figure 3.2.1. Real and imaginary parts of components of eigenvector $\boldsymbol{v}^{(M)}{}_I^{\#}(p)$

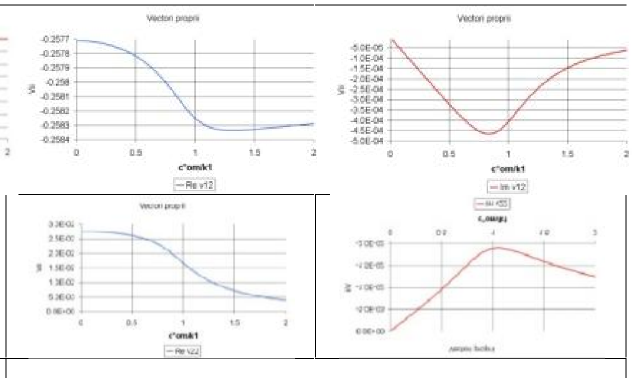

Figure 3.2.2. Real and imaginary parts of components of eigenvector $\boldsymbol{v}^{(M)}{}_{II}^{\#}(p)$