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ASPECTS CONCERNING THE EIGENVALUES DEPENDENCE ON THE DISKS THERMAL REGIME AND ROTATION

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Abstract: *From the analytical equations study of the disks behavior, some very important issues yield, issues which will be analyzed and solved in the following research.*
In the present paper one takes into account the fact that the study of the static and dynamical bending subjected disks, most be done mostly under the existence of a membrane stresses field (rotational motion, temperature field with a nonuniform distribution along the radius etc.). In the case of a plate of constant thickness, one uses the equation below:

$$D\Delta^2 w + \zeta(w) + \rho h \frac{\partial^2 w}{\partial t^2} = q \qquad (1)$$

where: w – the displacement measured normal to the plate's plane;
q – the distributed force, normal to the plate's plane;
$D = \frac{Eh^3}{12(1-\nu^2)}$ *- cylindrical modulus of elasticity, at bending of the plate;*
Δ *- Laplace's operator;*
ν *- Poisson's ratio;*
h – the thickness of the plate;
ρ *- the material density.*

Keywords: *Disk, eigenvalues problems, membrane stresses, critical stresses, stability.*

One mentions that the operator ζ has the following expression in polar coordinates:

$$\zeta(w) = -h\left[\sigma_r \frac{\partial^2 w}{\partial r^2} + \sigma_\theta\left(\frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2 \partial \theta^2}\right) + 2\tau_{r\theta}\frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial w}{\partial \theta}\right) - R\cdot\frac{\partial w}{\partial r} - \Theta\frac{\partial w}{r\partial \theta}\right] \qquad (2)$$

One used also the notations: σ_r, σ_θ, $\tau_{r\theta}$ for membrane stresses, and R and θ - massic forces as volume unit acting in the plate's plane. If the membrane stresses do not depend on the load normal to the plate's plane, the equation (1) becomes linear depending on the displacements w.
By applying the FEM, the partial derivatives equation (1) is replaced by the system of linear equations:

$$([K]+[K_G])\{u\}+[M]\{\ddot{u}\}=\{P\}, \qquad (3)$$

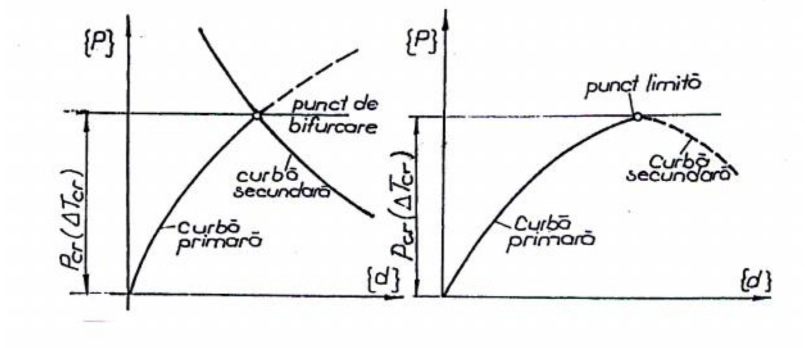
where: [K] – the elastic stiffness matrix;
$[K_G]$ – the geometrical stiffness matrix;
[M] – the masses matrix;
{u}; $\{\ddot{u}\}$ - vectors of nodal displacements and of nodal accelerations, respectively;
{P} – vector of nodal load.

One mentions that the solution of the problem given by equation (3) may be found in any easy way, if one performs the resolution of the displacement vector {u} in the eigenvectors space.
Based on the above mentioned issues, one can set two eigenvalues problems.
The first eigenvalues problem is given by the relations:

$$([K]+\lambda[K_G])\{u\}=0 \qquad (4)$$

Equation (4) gives the critical value of the membrane stresses, when the loss of stability due to branching off occurs and also the plate's configuration according to which buckling occurs.

The change of the structure's type of equilibrium is given by the critical load and the corresponding point is called critical point.
These changes are governed by the stability theorems stated by Thompson in 1970.
1. A primary curve, initially stable, which grows monotonically with the load, cannot become unstable without cutting another curve, called secondary curve, different from the primary one.
2. A primary curve, initially stable, which grows monotonically with the load, cannot have in the critical point an unstable equilibrium but if another equilibrium curve exists in its neighborhood (which can be in the primary curve extension), at loads values below the critical ones, fig 1, b.



a. b.

Fig. 1

The eigenvalues and the eigenvectors of problem (4) will be denoted by λ_i and $\{v_i]$, λ being a likeness coefficient, related to the stresses level of the solution of problem (4).
The second eigenvalues problem is written as follows:

$$\left([K]-\mu^2[M]\right)\{u\}=0 \tag{5}$$

The solution of this eigenvalues problem allows find out the eigenfrequencies and the eigenmodes of plates vibration. The eigenvalues and the eigenvectors of problem (5) will be denoted by μ_i^2 and $\{w_i\}$, respectively.
Further on one sets the problem of eigenmodes frequencies determination, for a plane plate, in the presence of membrane stresses fields.
One mainly follows to find the solution of the eigenvalues problem given by equation (3), which is written as:

$$\left([K]+\alpha[K_G]-\mu^2[M]\right)\{u\}=0 \tag{6}$$

starting with the solutions of the eigenvalues problems, given by equations (4) and (5).
The eigenvalues and eigenvectors of problem (6) will be denoted by Ω_i and $\{S_i\}$, respectively. One considers that the mentioned eigenvalues λ_i are different (distinct) and that is the reason why one can resolve the vectors $\{S_i\}$ after the vectors $\{V_i\}$, with the equations:

$$\{S_i\}=\sum b_i^j\{V_j\} \tag{7}$$

where b_i^j are unknown coefficients. By replacing equation (7) into (6), yields:

$$[K]\sum_j\left(1-\frac{\alpha}{\alpha_i}\right)b_i^j\{V_j\}=\Omega_i^2[M]\sum_j b_i^j\{V_j\} \tag{8}$$

One amplifies equation (8) at left with $\{V_k\}^T$ successively, where k = 1, 2, …, N, N being the number of nodal unknowns, that means the number of eigenvalues and eigenvectors. Taking into account that the eigenvectors listen to the orthogonality properties, one gets the relation:

$$\{V_k\}^T[K]\{V_k\}\left(1-\frac{\alpha}{\alpha_k}\right)b_i^k=\Omega_i^2\{V_k\}^T[M]\sum_j b_i^j\{V_j\} \tag{9}$$

where k = 1,2, …, N.
It is easy to observe that equation (9) represents a system of linear equations for the solution of the unknown b_i^j with j = 2,…,N, and Ω_i^2; one mentions that $b_1' = 1$. One establishes that one can write N such systems for each eigenvector $\{S_i\}$.

As yield from relation (9), if the membrane stresses achieve the critical values (like in any elastic stability calculation, it makes sense to retain only the smallest critical value, denoted by λ_k), than from the equation with number k we get $\Omega_i^2 = 0$, because the vectors $\{V_i\}$ are not orthogonal and the matrix [M] is positive definite.

Generally, if the membrane stresses increase and tend to the critical value, than the eigenfrequencies decrease and tend to zero.
One repeats theoretically the most common cases met in practice of bending subjected circular plates, in the presence of the axisymmetrical membrane stresses fields. The axisymmetrical membrane stresses fields are yield from the centrifugal force or can be due to the nonuniform temperature distribution along the radius or even due to a pre-stressing etc. For these multiple and frequent situations, the displacement is resolved in trigonometric series and one makes a separation of variables:

$$W(r,^{-}) = W_0(r) + \sum_{n=1}^{\infty} W_n(r) \cdot \cos n^{-} \tag{10}$$

In this way of variable separation, one decouples the system of equations (11), the unknowns of cach system referring to eigenmodes with the same number of nodal diameters. The nodal lines are, in this case, nodal circles and nodal diameters, no matter if the talk about the loss of stability or nodal vibrations.

$$\sim(\ W_0) = (\ -1)q - \ h\left[\pm_r \frac{\partial^2 w}{\partial r^2} + \pm_{-}\left(\frac{\partial w}{r\partial r} + \frac{\partial^2 w}{r^2 \partial^{-2}}\right) + 2^2{}_{r^{-}} \frac{\partial}{\partial r}\left(\frac{1}{r}\cdot\frac{\partial w}{\partial^{-}}\right) - R\frac{\partial w}{\partial r} - \Theta \frac{\partial w}{r\partial^{-}}\right] =$$
$$= (\ -1)q + \ \frac{h}{D} G(W_0) \tag{11}$$

Equation (11) was studied in another paper.
One establishes that of practical importance are, especially the eigenmodes with zero nodal circles, because under these circumstances the less critical stresses and the minimum eigenfrequencies yield.
One observes that the eigenmodes with zero nodal circles, for a certain number of nodal diameters change extremely less from problem to problem. Based on this last observation, one may write:

$$\{V_i\} \cong \{w_i\} \cong \{S_i\} \tag{12}$$

With this ultimate notice, the system of equations (9) is reduced to a single one, yielding:

$$\Omega_i^2 = \frac{\{V_i\}^T [K]\{V_i\}}{\{V_i\}^T [M]\{V_i\}} \cdot \left(1 - \frac{a}{a_i}\right) = \mu_i^2 \left(1 - \frac{a}{a_i}\right)$$

and finally:

$$\Omega_i = \mu_i \sqrt{1 - \frac{a}{a_i}} \tag{13}$$

equations which leads to a parabolic dependence between the eigenvalues Ω_i of the eigenvalues problem given by equation (6) and the eigenvalues ω_i of the eigenvalues problem described by relation (5). This solution is in good concordance with the precise theoretical measurement.
The extension of the theory presented in the present paragraph may be based on an analogue problem to the one studied until now.
It consists in the determination of the critical stresses of loss of stability, in the presence of two or more superposed membrane stresses fields. One considers that the critical stressed are known for each membrane stress field partly.
One leads to the solution of the eigenvalues problem under the generalized form:

$$\left([K] + a\sum_n [K_{Gn}]\right)\{u\} = 0 \tag{14}$$

where the eigenvalues and eigenvectors are known for the particular problems of eigenvalues.

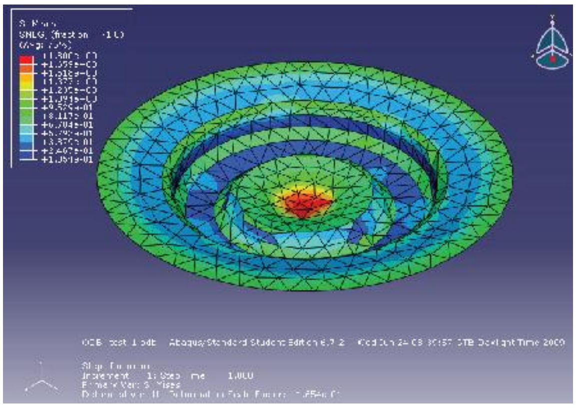
$$([K] + a[K_G])\{u\} = 0 \tag{15}$$

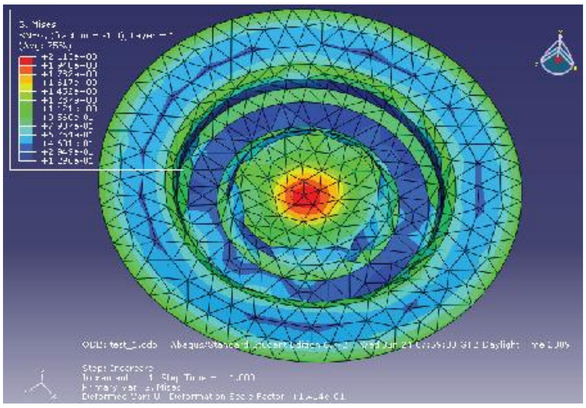
In the case of circular plates, reasoning as above, for modes with the zero nodal circles, the loss of stability occurs when the following equation is accomplished:

$$\sum \frac{a^n}{a_i^n} = 1 \tag{16}$$

where λ^n is the magnitude of the stresses field denoted by n, corresponding to the value which solved the eigenvalues problem denoted by (15) and λ_i – the eigenvalues corresponding to a certain modulus i. One notices a linear dependence between the critical values of the membrane stresses field, which are part of the load.
According to equation (16) if one considers that the disk is subjected simultaneous both the membrane stresses field due to the rotational motion of the disk and to the non-uniform distribution of the temperature along the radius, one may write:

$$\frac{a^T}{a^T_{cr}} + \frac{a^R}{a^R_{cr}} = 1 \qquad (17)$$





which demonstrates the above mentioned linear dependence .

By adding the solution presented in thi $\Omega_i = \mu_i \sqrt{1 - \sum_n \frac{a^n}{a^n_i}}$ s paper, one finally generalizes equation (13), as follows:

$$\Omega_i = \mu_i \sqrt{1 - \sum_n \frac{a^n}{a^n_i}} \qquad (18)$$

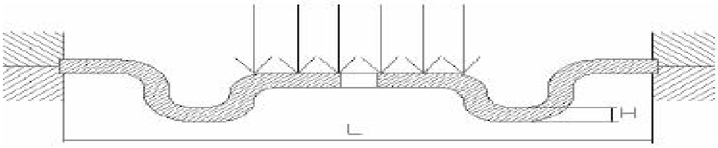


Figure 10. Sectioned rubber membrane.

The solutions obtained in this paper are very useful, because they allow get quickly new quantities (eigenfrequencies, critical stresses) without performing complicated calculations.
Based on the theoretically and experimentally results, other useful conditions yield too. For instance, if the eigenvalue λ_i is positive, than the eigenfrequency decreases and if λ_i is negative, the eigenfrequency increases.

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