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GREEN FUNCTION MATRICES FOR ELASTICALLY RESTRAINED HETEROGENEOUS CURVED BEAMS

György Szeidl[1], **László Kiss**[1]
[1]Institute of Applied Mechanics, University of Miskolc, Miskolc, HUNGARY
gyorgy.szeidl@uni-miskolc.hu, mechkiss@uni-miskolc.hu

Abstract: *Vibrations of heterogeneous curved beams have recently been investigated* [1, 2, 3]. *The thesis and papers cited deal with, among others, the issue what effect the central load has on the vibration of pinned-pinned and fixed-fixed curved beams made of heterogeneous material. To find numerical solution, the authors determine the Green function matrices for pre-loaded beams. Then they reduce the eigenvalue problems, which yield the eigenfrequencies as a function of the load, to eigenvalue problems governed by a system of Fredholm integral equations. A similar investigation for elastically restrained heterogeneous curved beams is provided in this article. The end-restraints are modeled by linear volute springs.*

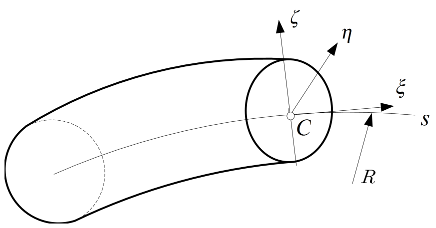
Keywords: *Curved beams, functionally graded materials (FGM), rotational restraint, natural frequency as a function of the load, Green function matrix*

1. INTRODUCTION

Curved beams are widely used in engineering applications – let us consider, for instance, arch bridges or roof structures. Research concerning the mechanical behavior of such structural elements began in the 19^{th} century. The free vibrations of curved beams have been under extensive investigation – see, e.g.: [4, 5] for more details.

Considering the vibrations of pre-loaded circular beams, the number of the available articles is much less than for the free vibrations. Wasserman [6], for example, investigates the load-frequency relationship for spring supported inextensible arches. The load can be dead or follower. Here, similarly as in [7], the Galerkin method was presented as an effective way to get solutions. Chidamparam and Leissa [7] investigate the vibrations of pinned-pinned and fixed-fixed prestressed homogeneous circular arches under distributed loads. The extensibility of the centerline is taken into account. We intend to contribute to the literature by improving the mechanical model, extending it for heterogeneous materials and using a numerical technique based on the Green function matrix.

2. KINEMATICAL ASSUMPTIONS & GOVERNING EQUATIONS



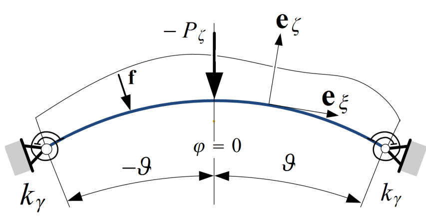


Figure 1. (a) Coordinate system (b) 1D model of the curved beam

We have developed a 1D beam model to investigate the vibratory problem. The curvilinear coordinate system ($\xi = s, \eta, \zeta$) is attached to the ($E-$weighted) centerline as shown in Figure 1. The radius of curvature R is constant there and, moreover, the cross-section geometry and material distribution are uniform. However, the material composition and thus the material parameters can vary over the cross-sectional coordinates η, ζ as long as axis ζ is a symmetry axis both for the geometry and for the material distribution. Therefore, it is possible to model homogeneous, functionally graded (FG) and even multi-layered beams, considering each material component to be linearly elastic and isotropic.

The centerline intersects the shown cross-section at C, whose location can be found by fulfilling the definition that the E-weighted first moment of the cross-section with respect to the axis η is zero there:

$$Q_{e\eta} = \int_A E(\eta,\zeta)\,\zeta\,\mathrm{d}A = 0\,. \tag{1}$$

We consider the validity of the Euler-Bernoulli beam theory for the investigations, i.e., the cross-sections rotate as if they were rigid bodies and remain perpendicular to the deformed centerline. Let u_o, w_o and ϑ be the tangential, radial displacement coordinates and the semi-vertex angle of the beam. Since the radius is constant the coordinate line s and the angle coordinate φ are related to each other by $s = R\varphi$. The axial strain $\varepsilon_{o\xi}$ and the rigid body rotation $\psi_{o\eta}$ on the centerline can be expressed [1] in terms of the displacements as

$$\varepsilon_{o\xi} = \frac{\mathrm{d}u_o}{\mathrm{d}s} + \frac{w_o}{R}, \qquad \psi_{o\eta} = \frac{u_o}{R} - \frac{\mathrm{d}w_o}{\mathrm{d}s}\,. \tag{2}$$

The principle of virtual work for the beam shown in Figure 1 (b) yields equilibrium equations

$$\frac{\mathrm{d}N}{\mathrm{d}s} + \frac{1}{R}\left[\frac{\mathrm{d}M}{\mathrm{d}s} - \left(N + \frac{M}{R}\right)\psi_{o\eta}\right] + f_t = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}M}{\mathrm{d}s} - \left(N + \frac{M}{R}\right)\psi_{o\eta}\right] - \frac{N}{R} + f_n = 0 \tag{3a}$$

which should be fulfilled by the axial force N and the bending moment M. Here f_t and f_n denote the intensity of the distributed loads in the tangential and normal directions ($\mathbf{f} = f_t\mathbf{e}_t + f_n\mathbf{e}_n$).

Recalling Hooke's law [8], the relation between the strains and inner forces become

$$N = \frac{I_{e\eta}}{R^2}\varepsilon_{o\xi} - \frac{M}{R}\,, \qquad M = -I_{e\eta}\left(\frac{\mathrm{d}^2 w_o}{\mathrm{d}s^2} + \frac{w_o}{R^2}\right)\,, \quad N + \frac{M}{R} = \frac{I_{e\eta}}{R^2}\varepsilon_{o\xi}\,, \quad \text{where} \tag{4}$$

$$A_e = \int_A E(\eta,\zeta)\mathrm{d}A\,, \qquad I_{e\eta} = \int_A E(\eta,\zeta)\zeta^2\mathrm{d}A\,, \qquad m = \frac{A_e R^2}{I_{e\eta}} - 1 \tag{5}$$

A_e is the E-weighted area of the cross-section, $I_{e\eta}$ is the E-weighted moment of inertia with respect to the bending axis while m is a geometry-heterogeneity parameter – the effect of the material distribution is incorporated into the model through the latter one. For simplicity reasons, we introduce dimensionless displacements and a notational convention for the derivatives taken with respect to the angle coordinate:

$$U_o = \frac{u_o}{R}\,, \qquad W_o = \frac{w_o}{R}\,; \qquad (\ldots)^{(n)} = \frac{\mathrm{d}^n(\ldots)}{\mathrm{d}\varphi^n}\,, \quad n \in \mathbb{Z}\,. \tag{6}$$

If we plug equations (2) and (4) into (3) and perform some manipulations – these are detailed in [1] – we get:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 - m\varepsilon_{o\xi} \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(2)} + \\ + \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 + m(1 - \varepsilon_{o\xi}) \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix} = \frac{R^3}{I_{e\eta}}\begin{bmatrix} f_t \\ f_n \end{bmatrix}\,. \tag{7}$$

Within the framework of the linear theory, we can freely neglect the effect of the deformations on the equilibrium (i.e., $\varepsilon_{o\xi} = 0$).

In the sequel, the increments (which occur because of the vibratory nature of the problem) in the typical quantities are identified by a subscript $_b$. Each physical quantity can be given in a form similar to the total tangential displacement which is equal to the sum $u_o + u_{ob}$. Here u_o is the static displacement caused by the pre-load, and u_{ob} is the dynamic displacement increment.

It turns out that the increments in the axial strain and in the rotation have a similar structure to equations (2):

$$\varepsilon_{mb} = \varepsilon_{o\xi\,b} + \psi_{o\eta}\psi_{o\eta\,b} \simeq \varepsilon_{o\xi\,b},\,, \qquad \psi_{o\eta\,b} = \frac{u_{ob}}{R} - \frac{\mathrm{d}w_{ob}}{\mathrm{d}s}\,, \qquad \varepsilon_{o\xi\,b} = \frac{\mathrm{d}u_{ob}}{\mathrm{d}s} + \frac{w_{ob}}{R}\,. \tag{8}$$

The principle of virtual work for the increments yields the equilibrium equations – see [1] for details [1]:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(N_b + \frac{M_b}{R}\right) - \frac{1}{R}\left(N + \frac{M}{R}\right)\psi_{o\eta\,b} + f_{tb} = 0\,, \tag{9a}$$

$$\frac{\mathrm{d}^2 M_b}{\mathrm{d}s^2} - \frac{N_b}{R} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\left(N + \frac{M}{R}\right)\psi_{o\eta\,b} + \left(N_b + \frac{M_b}{R}\right)\psi_{o\eta}\right] + f_{nb} = 0, \tag{9b}$$

where f_{tb} and f_{nb} are forces of inertia:

$$f_{tb} = -\rho_a A\frac{\partial^2 u_{ob}}{\partial t^2}, \qquad f_{nb} = -\rho_a A\frac{\partial^2 w_{ob}}{\partial t^2}\,, \tag{10}$$

[1]Thesis [1] is downloadable from the url address http://www.siphd.uni-miskolc.hu/ertekezesek/2015/KissLaszloPeter_phd.pdf

in which A as the area of the cross-section and ρ_a as the average density on the cross-section. The increments of the inner forces can be given in terms of the displacement increments via Hooke's law:

$$N_b = \frac{I_{e\eta}}{R^2} m\varepsilon_{o\xi\, b} - \frac{M_b}{R}\,, \qquad M_b = -I_{e\eta}\left(\frac{\mathrm{d}^2 w_{ob}}{\mathrm{d}s^2} + \frac{w_{ob}}{R^2}\right), \qquad N_b + \frac{M_b}{R} = \frac{I_{e\eta}}{R^2} m\varepsilon_{o\xi\, b}\,. \tag{11a}$$

Substituting (8) and (11) into (9), we get the equilibrium equations in the following form:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 - m\varepsilon_{o\xi} \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix}^{(2)} +$$

$$+ \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 + m\,(1 - \varepsilon_{o\xi}) \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix} = \frac{R^3}{I_{e\eta}} \begin{bmatrix} f_{tb} \\ f_{nb} \end{bmatrix}. \tag{12}$$

As regards the details we refer the reader to [1]. For harmonic vibrations the amplitudes $\hat{U}_{ob}$ and $\hat{W}_{ob}$ should satisfy the following differential equations:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 - m\varepsilon_{o\xi} \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(2)} +$$

$$+ \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 + m\,(1 - \varepsilon_{o\xi}) \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix} = \lambda \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}; \quad \lambda = \rho_a A \frac{R^3}{I_{e\eta}} \alpha^2\,, \tag{13}$$

where λ and α denote the eigenvalues and eigenfrequencies. The left side of (13) can be rewritten as

$$\mathbf{K}\left[\mathbf{y}\left(\varphi\right), \varepsilon_{o\xi}\right] = \overset{4}{\mathbf{P}}\mathbf{y}^{(4)} + \overset{2}{\mathbf{P}}\mathbf{y}^{(2)} + \overset{1}{\mathbf{P}}\mathbf{y}^{(1)} + \overset{0}{\mathbf{P}}\mathbf{y}^{(0)}, \qquad \mathbf{y}^T = \left[\hat{U}_{ob} \mid \hat{W}_{ob}\right]. \tag{14}$$

Let $\mathbf{r}(\varphi)$ be a dimensionless load: $\mathbf{r}^T = [r_1 \mid r_2]$. The differential equation

$$\mathbf{K}\left[\mathbf{y}\left(\varphi\right), \varepsilon_{o\xi}\right] = \mathbf{r} \tag{15}$$

describes the behavior of the pre-loaded beam if the beam is subjected to a further load $\mathbf{r}$ given here in a dimensionless form.

The free vibrations ($\varepsilon_{o\xi} = 0$) of heterogeneous circular beams are governed by the differential equations

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(2)} + \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & m+1 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix} = \lambda \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}. \tag{16}$$

For rotationally restrained beams differential equations (13), (15) and (16) are associated with the following boundary conditions:

$$\hat{U}_{ob}\Big|_{\varphi=\pm\vartheta} = 0, \qquad \hat{W}_{ob}\Big|_{\varphi=\pm\vartheta} = 0, \qquad \hat{W}_{ob}^{(2)} \pm \mathcal{K}_\gamma \hat{W}_{ob}^{(1)}\Big|_{\varphi=\pm\vartheta} = 0 \tag{17}$$

where $\mathcal{K}_\gamma = k_\gamma R / I_{e\eta}$ is a dimensionless spring constant given in terms of k_γ the spring constant. Equations (13), (17) [(14) and (17)] determine an eigenvalue problem. The i-th eigenfrequency α_i depends on the heterogeneity parameters m and ρ_a; and also on the magnitude and the direction of the concentrated force P_ζ. The effect of the latter one is accounted through the axial strain it causes: $\varepsilon_{o\xi} = \varepsilon_{o\xi}(\mathcal{P}, m, \vartheta)$. Here $\mathcal{P}$ is a dimensionless load: $\mathcal{P} = P_\zeta R^2 \vartheta / (2 I_{e\eta})$.

3. Numerical solution algorithm

The definition and the most important properties of the Green function matrix can be found in [4, 1]. Solution to the inhomogeneous boundary value problem (15), (17) is sought in the form

$$\mathbf{y}(\varphi) = \int_{-\vartheta}^{\vartheta} \mathbf{G}(\varphi,\psi)\mathbf{r}(\psi)\mathrm{d}\psi\,, \quad \mathbf{G}(\varphi,\psi) = \begin{bmatrix} G_{11}(\varphi,\psi) & G_{12}(\varphi,\psi) \\ G_{21}(\varphi,\psi) & G_{22}(\varphi,\psi) \end{bmatrix}, \tag{18}$$

where $\mathbf{G}$ is the Green function matrix and φ, ψ are angle coordinates.

If we write $\lambda\mathbf{y}$ for $\mathbf{r}$ in (18), the eigenvalue problem (13), (17) is replaced by a homogeneous integral equation system:

$$\mathbf{y}(\varphi) = \lambda \int_{-\vartheta}^{\vartheta} \mathbf{G}(\varphi,\psi)\mathbf{y}(\psi)\mathrm{d}\psi\,. \tag{19}$$

Numerical solution to such problems can be sought e.g., by quadrature methods [9]. Consider the integral formula

$$J(\phi) = \int_{-\vartheta}^{\vartheta} \phi(\psi)\,\mathrm{d}\psi \equiv \sum_{j=0}^{n} w_j \phi(\psi_j), \qquad \psi_j \in [-\vartheta, \vartheta]\,, \tag{20}$$

where $\psi_j(\varphi)$ is a vector and w_j are the known weights. Having utilized the latter equation, we obtain from (19) that

$$\sum_{j=0}^{n} w_j \mathbf{G}(\varphi,\psi_j)\tilde{\mathbf{y}}(\psi_j) = \tilde{\kappa}\tilde{\mathbf{y}}(\varphi) \qquad \tilde{\kappa} = 1/\tilde{\lambda} \qquad \in [-\vartheta,\vartheta] \tag{21}$$

is the solution, which yields an approximate eigenvalue $\tilde{\lambda} = 1/\tilde{\kappa}$ and the corresponding approximate eigenfunction $\tilde{\mathbf{y}}(\varphi)$. After setting φ to ψ_i $(i = 0, 1, 2, \ldots, n)$, we have

$$\sum_{j=0}^{n} w_j \mathbf{G}(\psi_i,\psi_j)\tilde{\mathbf{y}}(\psi_j) = \tilde{\kappa}\tilde{\mathbf{y}}(\psi_i) \qquad \tilde{\kappa} = 1/\tilde{\lambda} \qquad \psi_i,\psi_j \in [-\vartheta,\vartheta]\,, \quad \text{or} \quad \mathcal{G}\mathcal{D}\tilde{\mathcal{Y}} = \tilde{\kappa}\tilde{\mathcal{Y}}\,, \tag{22}$$

where $\mathcal{G} = [\mathbf{G}(\psi_i,\psi_j)]$ for self-adjoint problems, while $\mathcal{D} = \operatorname{diag}(w_0,\ldots,w_0|\ldots|w_n,\ldots,w_n)$ and $\tilde{\mathcal{Y}}^T = [\tilde{\mathbf{y}}^T(\psi_0)|\tilde{\mathbf{y}}^T(\psi_1)|\ldots|\tilde{\mathbf{y}}^T(\psi_n)]$. After solving the generalized algebraic eigenvalue problem (22), we have the approximate eigenvalues $\tilde{\lambda}_r$ and eigenvectors $\mathcal{Y}_r$, while the corresponding eigenfunction is obtained via substituting into (21):

$$\tilde{\mathbf{y}}_r(\varphi) = \tilde{\lambda}_r \sum_{j=0}^{n} w_j \mathbf{G}(\varphi,\psi_j)\tilde{\mathbf{y}}_r(\psi_j) \qquad r = 0, 1, 2, \ldots, n\,. \tag{23}$$

Divide the interval $[-\vartheta,\vartheta]$ into equidistant subintervals of length h and apply the integration formula to each subinterval. By repeating the line of thought leading to (23), the algebraic eigenvalue problem obtained has the same structure as (23).

It is also possible to consider the integral equations (19) as if they were boundary integral equations and apply isoparametric approximation on the subintervals (elements). If this is the case, one can approximate the eigenfunction on the e-th element (the e-th subinterval which is mapped onto the interval $\gamma \in [-1,1]$ and is denoted by $\mathfrak{L}_e$) by

$$\overset{e}{\mathbf{y}} = \mathbf{N}_1(\gamma)\overset{e}{\mathbf{y}}_1 + \mathbf{N}_2(\gamma)\overset{e}{\mathbf{y}}_2 + \mathbf{N}_3(\gamma)\overset{e}{\mathbf{y}}_3\,, \tag{24}$$

where quadratic local approximation is assumed: $\mathbf{N}_i = \operatorname{diag}(N_i),\ N_1 = 0.5\gamma(\gamma-1),\ N_2 = 1-\gamma^2,\ N_3 = 0.5\gamma(\gamma+1),\ \overset{e}{\mathbf{y}}_i$ is the value of the eigenfunction $\mathbf{y}(\varphi)$ at the left endpoint, the midpoint and the right endpoint of the element, respectively. Upon substitution of approximation (24) into (19), we have

$$\tilde{\mathbf{y}}(\varphi) = \tilde{\lambda}\sum_{e=1}^{n_{be}} \int_{\mathfrak{L}_e} \mathbf{G}(x,\gamma)[\mathbf{N}_1(\eta)|\mathbf{N}_2(\gamma)|\mathbf{N}_3(\gamma)]d\gamma \left[\overset{e}{\mathbf{y}}_1|\overset{e}{\mathbf{y}}_2|\overset{e}{\mathbf{y}}_3\right]^T\,, \tag{25}$$

in which, n_{be} is the number of elements. Using equation (25) as a point of departure, and repeating the line of thought leading to (22), we get again an algebraic eigenvalue problem.

4. THE GREEN FUNCTION MATRICES

Based on theses [4, 1], the Green function can be given in the form

$$\underbrace{\mathbf{G}(\varphi,\psi)}_{(2\times 2)} = \sum_{j=1}^{4} \mathbf{Y}_j(\varphi)\left[\mathbf{A}_j(\psi) \pm \mathbf{B}_j(\psi)\right]\,, \tag{26}$$

where (a) the sign is [positive](negative) if $[\varphi \le \psi](\varphi \ge \psi)$; (b) the matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ have the following structure

$$\mathbf{A}_j = \begin{bmatrix} \overset{j}{A}_{11} & \overset{j}{A}_{12} \\ \overset{j}{A}_{21} & \overset{j}{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{j1} & \mathbf{A}_{j2} \end{bmatrix}\,, \qquad \mathbf{B}_j = \begin{bmatrix} \overset{j}{B}_{11} & \overset{j}{B}_{12} \\ \overset{j}{B}_{21} & \overset{j}{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{j1} & \mathbf{B}_{j2} \end{bmatrix} \quad j = 1,\ldots,4; \tag{27}$$

(c) the coefficients in $\mathbf{B}_j$ are independent of the boundary conditions. The columns in the matrices $\mathbf{Y}_i$ are solutions to the homogeneous differential equations $\mathbf{K}\left[\mathbf{y}(\varphi), \varepsilon_{o\xi}\right] = \mathbf{0}$. If $\varepsilon_{o\xi} < 0$ (the concentrated force is compressive), then

$$\mathbf{Y}_1 = \begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & 0 \end{bmatrix},\ \mathbf{Y}_2 = \begin{bmatrix} -\sin\varphi & 0 \\ \cos\varphi & 0 \end{bmatrix},\ \mathbf{Y}_3 = \begin{bmatrix} \cos\chi\varphi & \mathcal{M}\varphi \\ \chi\sin\chi\varphi & -1 \end{bmatrix},\ \mathbf{Y}_4 = \begin{bmatrix} -\sin\chi\varphi & 1 \\ \chi\cos\chi\varphi & 0 \end{bmatrix}. \tag{28}$$

However, $\mathbf{Y}_3$ and $\mathbf{Y}_4$ are different when $m\varepsilon_{o\xi} > 1$:

$$\mathbf{Y}_3 = \begin{bmatrix} \cosh\chi\varphi & \mathcal{M}\varphi \\ -\chi\sinh\chi\varphi & -1 \end{bmatrix},\ \mathbf{Y}_4 = \begin{bmatrix} \sinh\chi\varphi & 1 \\ -\chi\cosh\chi\varphi & 0 \end{bmatrix},\quad \mathcal{M} = \frac{m+1}{m(1+\varepsilon_{o\xi})}. \tag{29}$$

Consequently we should deal with these two loading cases separately.

The Green functions matrix if $\varepsilon_{o\xi} < 0$. Let us now introduce the following notational conventions

$$a = \overset{1}{B}_{1i},\ b = \overset{2}{B}_{1i},\ c = \overset{3}{B}_{1i},\ d = \overset{3}{B}_{2i},\ e = \overset{4}{B}_{1i},\ f = \overset{4}{B}_{2i}\,. \tag{30}$$

It follows from the structure of the solutions $\mathbf{Y}_\ell$, $(\ell = 1, \ldots, 4)$ that $\overset{1}{B}_{2i} = \overset{2}{B}_{2i} = \overset{1}{A}_{2i} = \overset{2}{A}_{2i} = 0$. The systems of equations for the unknowns $a, \ldots, f$ can be set up by fulfilling the continuity and discontinuity conditions the Green function matrix [4, 1] should meet if $\varphi = \psi$. Therefore, if $i = 1$, we have

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cos(\chi\psi) & \mathcal{M}\psi & -\sin(\chi\psi) & 1 \\ \sin\psi & \cos\psi & \chi\sin(\chi\psi) & -1 & \chi\cos(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi\sin(\chi\psi) & \mathcal{M} & -\chi\cos(\chi\psi) & 0 \\ \cos\psi & -\sin\psi & \chi^2\cos(\chi\psi) & 0 & -\chi^2\sin(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi^3\sin(\chi\psi) & 0 & -\chi^3\cos(\chi\psi) & 0 \\ -\cos\psi & \sin\psi & -\chi^4\cos(\chi\psi) & 0 & \chi^4\sin(\chi\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2m} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{31}$$

from where we get the constants as

$$\begin{aligned} a &= \overset{1}{B}_{11} = \frac{\chi^2}{(1-\chi^2)(1-\mathcal{M})\,m}\frac{\sin\psi}{2}, & b &= \overset{2}{B}_{11} = \frac{\chi^2}{(1-\chi^2)(1-\mathcal{M})\,m}\frac{\cos\psi}{2}, \\ c &= \overset{3}{B}_{11} = -\frac{\chi^2}{(1-\chi^2)(1-\mathcal{M})\,m}\frac{\sin\chi\psi}{2\chi^3}, & d &= \overset{3}{B}_{21} = -\frac{1}{2(1-\mathcal{M})\,m}, \\ e &= \overset{4}{B}_{11} = -\frac{1}{\chi(1-\chi^2)(1-\mathcal{M})\,m}\frac{\cos\chi\psi}{2}, & f &= \overset{4}{B}_{21} = \frac{1}{2}\mathcal{M}\frac{\psi}{m(1-\mathcal{M})}. \end{aligned} \tag{32}$$

If $i = 2$, then

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cos(\chi\psi) & \mathcal{M}\psi & -\sin(\chi\psi) & 1 \\ \sin\psi & \cos\psi & \chi\sin(\chi\psi) & -1 & \chi\cos(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi\sin(\chi\psi) & \mathcal{M} & -\chi\cos(\chi\psi) & 0 \\ \cos\psi & -\sin\psi & \chi^2\cos(\chi\psi) & 0 & -\chi^2\sin(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi^3\sin(\chi\psi) & 0 & -\chi^3\cos(\chi\psi) & 0 \\ -\cos\psi & \sin\psi & -\chi^4\cos(\chi\psi) & 0 & \chi^4\sin(\chi\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \tag{33}$$

is the equation system, the solution of which assumes the form

$$\begin{aligned} a &= \overset{1}{B}_{12} = \frac{1}{2}\frac{\cos\psi}{(1-\chi^2)}, & b &= \overset{2}{B}_{12} = -\frac{1}{2}\frac{\sin\psi}{(1-\chi^2)}, & c &= \overset{3}{B}_{12} = -\frac{1}{2}\frac{\cos\chi\psi}{(1-\chi^2)\,\chi^2}, \\ d &= \overset{3}{B}_{22} = 0, & e &= \overset{4}{B}_{12} = \frac{1}{2}\frac{\sin\chi\psi}{(1-\chi^2)\,\chi^2}, & f &= \overset{4}{B}_{22} = \frac{1}{2\chi^2}. \end{aligned} \tag{34}$$

Let $\boldsymbol{\alpha}$ be an arbitrary column matrix of size (2×1) The unknown scalars

$$\overset{1}{A}_{1i}(\psi),\ \overset{2}{A}_{1i}(\psi),\ \overset{3}{A}_{1i}(\psi),\ \overset{3}{A}_{2i}(\psi),\ \overset{4}{A}_{1i}(\psi),\ \overset{4}{A}_{2i}(\psi),\, i = 1, 2;\, \psi \in [-\vartheta, \vartheta]$$

in the matrices $\mathbf{A}_j$ can be determined from the condition that the product $\mathbf{G}(\varphi, \psi)\boldsymbol{\alpha}$ should satisfy the boundary conditions (17). This leads to the equation system

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \cos\chi\vartheta & -\mathcal{M}\vartheta & \sin\chi\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & \cos\chi\vartheta & \mathcal{M}\vartheta & -\sin\chi\vartheta & 1 \\ -\sin\vartheta & \cos\vartheta & -\chi\sin\chi\vartheta & -1 & \chi\cos\chi\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & \chi\sin\chi\vartheta & -1 & \chi\cos\chi\vartheta & 0 \\ \sin\vartheta - \mathcal{K}_\gamma\cos\vartheta & -\cos\vartheta - \mathcal{K}_\gamma\sin\vartheta & \chi^3\sin\chi\vartheta - \mathcal{K}_\gamma\chi^2\cos\chi\vartheta & 0 & -\chi^3\cos\chi\vartheta - \mathcal{K}_\gamma\chi^2\sin\chi\vartheta & 0 \\ \mathcal{K}_\gamma\cos\vartheta - \sin\vartheta & -\cos\vartheta - \mathcal{K}_\gamma\sin\vartheta & \mathcal{K}_\gamma\chi^2\cos\chi\vartheta - \chi^3\sin\chi\vartheta & 0 & -\chi^3\cos\chi\vartheta - \mathcal{K}_\gamma\chi^2\sin\chi\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta - c\cos\chi\vartheta + d\mathcal{M}\vartheta - e\sin\chi\vartheta - f \\ a\cos\vartheta - b\sin\vartheta + c\cos\chi\vartheta + d\mathcal{M}\vartheta - e\sin\chi\vartheta + f \\ a\sin\vartheta - b\cos\vartheta + c\chi\sin\chi\vartheta + d - e\chi\cos\chi\vartheta \\ a\sin\vartheta + b\cos\vartheta + c\chi\sin\chi\vartheta - d + e\chi\cos\chi\vartheta \\ a(-\sin\vartheta + \mathcal{K}_\gamma\cos\vartheta) + b(\cos\vartheta + \mathcal{K}_\gamma\sin\vartheta) + c\left(-\chi^3\sin\chi\vartheta + \chi^2\cos\chi\vartheta\right) + e\left(\chi^3\cos\chi\vartheta + \mathcal{K}_\gamma\chi^2\sin\chi\vartheta\right) \\ a(-\sin\vartheta + \mathcal{K}_\gamma\cos\vartheta) - b(\cos\vartheta + \mathcal{K}_\gamma\sin\vartheta) + c\left(-\chi^3\sin\chi\vartheta + \chi^2\cos\chi\vartheta\right) - e\left(\chi^3\cos\chi\vartheta + \mathcal{K}_\gamma\chi^2\sin\chi\vartheta\right) \end{bmatrix}. \tag{35}$$

The closed form solutions are presented in Appendix A.

Calculation of the Green functions matrix if $\varepsilon_{o\xi} > 0$ and $m\varepsilon_{o\xi} > 1$. Similarly as above, we get the following equations if $i = 1$

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cosh(\chi\psi) & \mathcal{M}\psi & \sinh(\chi\psi) & 1 \\ \sin\psi & \cos\psi & -\chi\sinh(\chi\psi) & -1 & -\chi\cosh(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & \chi\sinh(\chi\psi) & \mathcal{M} & \chi\cosh(\chi\psi) & 0 \\ \cos\psi & -\sin\psi & -\chi^2\cosh(\chi\psi) & 0 & -\chi^2\sinh(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi^3\sinh(\chi\psi) & 0 & -\chi^3\cosh(\chi\psi) & 0 \\ -\cos\psi & \sin\psi & -\chi^4\cosh(\chi\psi) & 0 & -\chi^4\sinh(\chi\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2m} \\ 0 \\ 0 \\ 0 \end{bmatrix} . \tag{36}$$

The solutions are as follows

$$\begin{gathered} a = \overset{1}{B}_{11} = -\frac{\chi^2}{(1+\chi^2)(1-\mathcal{M})m}\frac{\sin\psi}{2}, \qquad b = \overset{2}{B}_{11} = -\frac{\chi^2}{(1+\chi^2)(1-\mathcal{M})m}\frac{\cos\psi}{2}, \\ c = \overset{3}{B}_{11} = -\frac{1}{\chi(1+\chi^2)(1-\mathcal{M})m}\frac{\sinh\chi\psi}{2}, \quad d = \overset{3}{B}_{21} = -\frac{1}{2(1-\mathcal{M})m}, \\ e = \overset{4}{B}_{11} = \frac{1}{\chi(1+\chi^2)(1-\mathcal{M})m}\frac{\cosh\chi\psi}{2}, \quad f = \overset{4}{B}_{21} = \frac{1}{2(1-\mathcal{M})m}\mathcal{M}\psi . \end{gathered} \tag{37}$$

If $i = 2$

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cosh(\chi\psi) & \mathcal{M}\psi & \sinh(\chi\psi) & 1 \\ \sin\psi & \cos\psi & -\chi\sinh(\chi\psi) & -1 & -\chi\cosh(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & \chi\sinh(\chi\psi) & \mathcal{M} & \chi\cosh(\chi\psi) & 0 \\ \cos\psi & -\sin\psi & -\chi^2\cosh(\chi\psi) & 0 & -\chi^2\sinh(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi^3\sinh(\chi\psi) & 0 & -\chi^3\cosh(\chi\psi) & 0 \\ -\cos\psi & \sin\psi & -\chi^4\cosh(\chi\psi) & 0 & -\chi^4\sinh(\chi\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \tag{38}$$

is the equation system to be solved – compare it to (33) – and the solutions we have obtained are:

$$\begin{gathered} a = \overset{1}{B}_{12} = \frac{1}{2}\frac{\cos\psi}{(1+\chi^2)}, \quad b = \overset{2}{B}_{12} = -\frac{1}{2}\frac{\sin\psi}{(1+\chi^2)}, \qquad c = \overset{3}{B}_{12} = \frac{1}{2}\frac{\cosh\chi\psi}{(1+\chi^2)\chi^2} \\ d = \overset{3}{B}_{22} = 0, \qquad e = \overset{4}{B}_{12} = -\frac{1}{2}\frac{\sinh\chi\psi}{\chi^2(1+\chi^2)}, \quad f = \overset{4}{B}_{22} = -\frac{1}{2\chi^2} . \end{gathered} \tag{39}$$

By repeating the steps that resulted in (35) we obtain the following equation system for $\overset{1}{A}_{1i}, \dots, \overset{4}{A}_{2i}$:

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \cosh\chi\vartheta & -\mathcal{M}\vartheta & -\sinh\chi\vartheta & 1 \\ \cos\vartheta & -\sin\vartheta & \cosh\chi\vartheta & \mathcal{M}\vartheta & \sinh\chi\vartheta & 1 \\ -\sin\vartheta & \cos\vartheta & \chi\sinh\chi\vartheta & -1 & -\chi\cosh\chi\vartheta & 0 \\ \sin\vartheta & \cos\vartheta & -\chi\sinh\chi\vartheta & -1 & -\chi\cosh\chi\vartheta & 0 \\ \sin\vartheta - \mathcal{K}_\gamma\cos\vartheta & -\cos\vartheta - \mathcal{K}_\gamma\sin\vartheta & \chi^3\sinh\chi\vartheta + \mathcal{K}_\gamma\chi^2\cosh\chi\vartheta & 0 & -\chi^3\cosh\chi\vartheta - \mathcal{K}_\gamma\chi^2\sinh\chi\vartheta & 0 \\ \mathcal{K}_\gamma\cos\vartheta - \sin\vartheta & -\cos\vartheta - \mathcal{K}_\gamma\sin\vartheta & -\chi^3\sinh\chi\vartheta - \mathcal{K}_\gamma\chi^2\cosh\chi\vartheta & 0 & -\chi^3\cosh\chi\vartheta - \mathcal{K}_\gamma\chi^2\sinh\chi\vartheta & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{11} \\ \overset{2}{A}_{11} \\ \overset{3}{A}_{11} \\ \overset{3}{A}_{21} \\ \overset{4}{A}_{11} \\ \overset{4}{A}_{21} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta - c\cosh\chi\vartheta + d\mathcal{M}\vartheta + e\sinh\chi\vartheta - f \\ a\cos\vartheta - b\sin\vartheta + c\cosh\chi\vartheta + d\mathcal{M}\vartheta + e\sinh\chi\vartheta + f \\ a\sin\vartheta - b\cos\vartheta - c\chi\sinh\chi\vartheta + d + e\chi\cosh\chi\vartheta \\ a\sin\vartheta + b\cos\vartheta - c\chi\sinh\chi\vartheta - d - e\chi\cosh\chi\vartheta \\ a(-\sin\vartheta + \mathcal{K}_\gamma\cos\vartheta) + b(\cos\vartheta + \mathcal{K}_\gamma\sin\vartheta) - c\left(\chi^3\sinh\chi\vartheta + \mathcal{K}_\gamma\chi^2\cosh\chi\vartheta\right) + e\left(\chi^3\cosh\chi\vartheta + \mathcal{K}_\gamma\chi^2\sinh\chi\vartheta\right) \\ a(-\sin\vartheta + \mathcal{K}_\gamma\cos\vartheta) - b(\cos\vartheta + \mathcal{K}_\gamma\sin\vartheta) - c\left(\chi^3\sinh\chi\vartheta + \mathcal{K}_\gamma\chi^2\cosh\chi\vartheta\right) - e\left(\chi^3\cosh\chi\vartheta + \mathcal{K}_\gamma\chi^2\sinh\chi\vartheta\right) \end{bmatrix}$$

The closed form solutions are presented in Appendix B.

Assume that $\mathcal{K}_\gamma \to 0$. Then the limit $\lim_{\mathcal{K}_\gamma\to 0}\mathbf{G}(\varphi,\psi)$ yields the Green function matrix for pre-loaded pinned-pinned beams [3]. When $\mathcal{K}_\gamma \to \infty$, the limit $\lim_{\mathcal{K}_\gamma\to\infty}\mathbf{G}(\varphi,\psi)$ results in the Green function matrix for pre-loaded fixed-fixed beams [2].

5. Computational results

In this section we shall present the most important result of the computations only. Let $\varepsilon_{o\xi\,\mathrm{crit}}$ be the axial strain that belongs to the load $\mathcal{P}_\zeta = \mathcal{P}_{crit}$ where $\mathcal{P}_{crit}$ is the critical load that causes the stability loss of the beam. Further let α_i be the i-th eigenfrequency of the loaded beam, while the eigenfrequencies that belong to the free vibrations (then the beam is unloaded) are denoted by $\alpha_{i\,\mathrm{free}}$.

Figure 2 represents the quotient $\alpha_1^2/\alpha_{1\,\mathrm{free}}^2$ against the quotient $|\varepsilon_{o\xi}/\varepsilon_{o\xi\,\mathrm{crit}}|$ both for a negative and for a positive P_ζ. We remark that this time the subscript 1 always refers to the lowest eigenfrequency (for small ϑ the order of the eigenfrequencies will change [4]). The frequencies under [compression] <tension> [decrease] <increase> almost linearly and independently of m, ϑ and $\mathcal{K}_\gamma$ given that $m >\sim 10\,000$ and $\bar{\vartheta} >\sim 1$. These relationships can be approximated with a very good accuracy by the linear functions

$$\frac{\alpha_1^2}{\alpha_{1\text{ free}}^2} = 1.000 - 0.9840 \frac{|\varepsilon_{o\xi}|}{\varepsilon_{o\xi\ crit}}\ , \quad \text{if} \quad \varepsilon_{o\xi} < 0\ , \tag{40}$$

$$\frac{\alpha_1^2}{\alpha_{1\text{ free}}^2} = 1.000 + 0.9860 \frac{|\varepsilon_{o\xi}|}{\varepsilon_{o\xi\ crit}}\ , \quad \text{if} \quad \varepsilon_{o\xi} > 0. \tag{41}$$

Note that these results are almost the same as those valid for pinned-pinned or fixed-fixed curved beams.

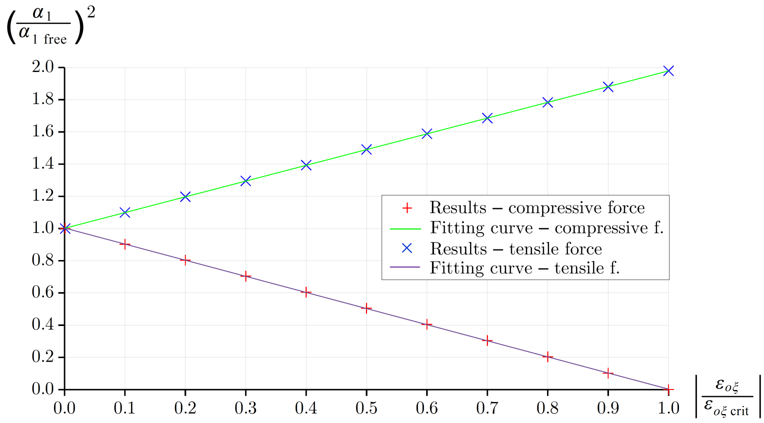


Figure 2. Results for the two loading cases of elastically restrained curved beams [2]

6. Concluding remarks

We have presented a new model to clarify the vibratory behavior of circular beams pre-loaded by central load (a vertical force at the crown point). The model is based on the Euler-Bernoulli beam theory and is applicable for heterogeneous materials. The beam-end supports are rotationally restrained pins, which are modeled by linear volute springs having the same spring constant. The effect of the pre-load is incorporated into the model via the strain it causes. An eigenvalue problem was established by using the principle of virtual work. This eigenvalue problem was transformed to an eigenvalue problem governed by Fredholm integral equations. The Green function matrix is given in a closed form both for compressive load and for tensile load. A numerical algorithm is proposed for the solution. Though some computational results are provided, further computations are needed to refine the results and to clarify how the higher eigenfrequencies depend on the load.

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Appendix A. Solutions for $\overset{1}{A}_{1i}, \ldots, \overset{4}{A}_{2i}$ if $\varepsilon_{o\xi} < 0$

Let us introduce the following constants:

$$\mathcal{D}_1 = -\sin\chi\vartheta\sin\vartheta + (\sin\chi\vartheta)\,K\cos\vartheta + \chi^2\sin\chi\vartheta\sin\vartheta - K\chi\cos\chi\vartheta\sin\vartheta = -\left(1-\chi^2\right)\sin\vartheta\sin\chi\vartheta + \mathcal{K}_\gamma\left(\cos\vartheta\sin\chi\vartheta - \chi\sin\vartheta\cos\chi\vartheta\right)\ , \tag{42}$$

and

$$\mathcal{D}_2 = \cos\vartheta\sin\chi\vartheta - \chi^3\sin\vartheta\cos\chi\vartheta - M\chi\vartheta\left(1-\chi^2\right)\cos\vartheta\cos\chi\vartheta + \mathcal{K}_\gamma\left[\left(1-\chi^2\right)\sin\vartheta\sin\chi\vartheta + M\chi\vartheta\left(\chi\cos\vartheta\sin\chi\vartheta - \sin\vartheta\cos\chi\vartheta\right)\right]\ . \tag{43}$$

Making use of the constants introduced the solutions sought can be given in the following forms:

$$\overset{1}{A}_{1i} = \overset{1}{A}_{1in}/\mathcal{D}_1, \quad \overset{1}{A}_{1in} = b\left(1-\chi^2\right)\cos\vartheta\sin\chi\vartheta + d\chi^2\sin\chi\vartheta + \mathcal{K}_\gamma\left[b\left(\sin\vartheta\sin\chi\vartheta + \chi\cos\vartheta\cos\chi\vartheta\right) - d\cos\chi\vartheta + e\chi^2\right]\ , \tag{44a}$$

$$\begin{aligned}\overset{2}{A}_{1i} = \overset{2}{A}_{1in}/\mathcal{D}_2, \quad \overset{2}{A}_{1in} &= a\chi^3\cos\vartheta\cos\chi\vartheta - aM\chi\vartheta\left(1-\chi^2\right)\sin\vartheta\cos\chi\vartheta + a\sin\vartheta\sin\chi\vartheta + c\chi^3 + f\chi^3\cos\chi\vartheta + \\ &\mathcal{K}_\gamma\left[aM\chi\vartheta\left(\chi\sin\vartheta\sin\chi\vartheta + \cos\vartheta\cos\chi\vartheta\right) - a\left(1-\chi^2\right)\cos\vartheta\sin\chi\vartheta + cM\chi^3\vartheta + f\chi^2\sin\chi\vartheta\right]\ ,\end{aligned} \tag{44b}$$

$$\overset{3}{A}_{1i} = \overset{3}{A}_{1in}/\chi\mathcal{D}_1, \quad \overset{3}{A}_{1in} = -d\sin\vartheta + e\chi\left(1-\chi^2\right)\sin\vartheta\cos\chi\vartheta - \mathcal{K}_\gamma\left[b + e\chi\left(\chi\sin\chi\vartheta\sin\vartheta + \cos\vartheta\cos\chi\vartheta\right) - d\cos\vartheta\right]\ , \tag{44c}$$

$$\begin{aligned}\overset{3}{A}_{2i} = \overset{3}{A}_{2in}/\mathcal{D}_2, \quad \overset{3}{A}_{2in} &= -\chi\left(1-\chi^2\right)\left(a\cos\chi\vartheta + c\cos\vartheta + f\cos\vartheta\cos\chi\vartheta\right) + \\ &+ \mathcal{K}_\gamma\left[-a\left(1-\chi^2\right)\sin\chi\vartheta - c\chi\left(1-\chi^2\right)\sin\vartheta - f\chi\left(\sin\vartheta\cos\chi\vartheta - \chi\cos\vartheta\sin\chi\vartheta\right)\right]\ ,\end{aligned} \tag{44d}$$

$$\overset{4}{A}_{1i} = -\overset{4}{A}_{1in}/\mathcal{D}_2, \quad \overset{4}{A}_{1in} = a + cM\vartheta\chi\left(1-\chi^2\right)\cos\vartheta\sin\chi\vartheta + c\left(\chi^3\sin\vartheta\sin\chi\vartheta + \cos\chi\vartheta\cos\vartheta\right) + f\cos\vartheta +$$
$$+\,\mathcal{K}_\gamma\left[aM\vartheta + cM\vartheta\chi\left(\chi\cos\vartheta\cos\chi\vartheta + \sin\vartheta\sin\chi\vartheta\right) + c\left(1-\chi^2\right)\sin\vartheta\cos\chi\vartheta + f\sin\vartheta\right], \tag{44e}$$

$$\overset{4}{A}_{2i} = -\overset{4}{A}_{2in}/\chi D_1, \quad \overset{4}{A}_{2in} = \chi b\left(1-\chi^2\right)\sin\chi\vartheta - dM\vartheta\chi\left(1-\chi^2\right)\sin\vartheta\sin\chi\vartheta + d\chi^3\cos\vartheta\sin\chi\vartheta - d\sin\vartheta\cos\chi\vartheta + e\chi\left(1-\chi^2\right)\sin\vartheta -$$
$$-\,\mathcal{K}_\gamma\left[b\left(1-\chi^2\right)\cos\chi\vartheta - d\left(1-\chi^2\right)\cos\vartheta\cos\chi\vartheta + dM\vartheta\chi\left(\chi\sin\vartheta\cos\chi\vartheta - \cos\vartheta\sin\chi\vartheta\right) + e\chi\left(1-\chi^2\right)\cos\vartheta\right]. \tag{44f}$$

APPENDIX B. SOLUTIONS FOR $\overset{1}{A}_{1i}, \ldots, \overset{4}{A}_{2i}$ IF $\varepsilon_{o\xi} > 0$ AND $m\varepsilon_{o\xi} > 1$.

Let

$$D_1 = \left(\chi^2+1\right)\sin\vartheta\sinh\chi\vartheta + \mathcal{K}_\gamma\left(\chi\sin\vartheta\cosh\chi\vartheta - \cos\vartheta\sinh\chi\vartheta\right) \tag{45}$$

and

$$D_2 = -\cos\vartheta\sinh\chi\vartheta - \chi^3\sin\vartheta\cosh\chi\vartheta + M\vartheta\chi\left(\chi^2+1\right)\cos\vartheta\cosh\chi\vartheta + \mathcal{K}_\gamma\left[-\left(\chi^2+1\right)\sin\vartheta\sinh\chi\vartheta + M\vartheta\chi\left(\chi\cos\vartheta\sinh\chi\vartheta + \sin\vartheta\cosh\chi\vartheta\right)\right]. \tag{46}$$

With the two constants introduced

$$\overset{1}{A}_{1i} = \overset{1}{A}_{1in}/D_1, \quad \overset{1}{A}_{1in} = b\left(\chi^2+1\right)\cos\vartheta\sinh\chi\vartheta - d\chi^2\sinh\chi\vartheta + \mathcal{K}_\gamma\left[b\left(\sin\vartheta\sinh\chi\vartheta + \chi\cos\vartheta\cosh\chi\vartheta\right) - d\chi\cosh\chi\vartheta - e\chi^2\right], \tag{47a}$$

$$\overset{2}{A}_{1i} = \overset{2}{A}_{1in}/D_2, \quad \overset{2}{A}_{1in} = -a\sin\vartheta\sinh\chi\vartheta + a\chi^3\cos\vartheta\cosh\chi\vartheta + aM\vartheta\chi\left(\chi^2+1\right)\sin\vartheta\cosh\chi\vartheta + f\chi^3\cosh\chi\vartheta + c\chi^3 +$$
$$+\,\mathcal{K}_\gamma\left[aM\vartheta\chi\left(\chi\sin\vartheta\sinh\chi\vartheta - \cos\vartheta\cosh\chi\vartheta\right) + c\chi^3 M\vartheta + a\left(\chi^2+1\right)\cos\vartheta\sinh\chi\vartheta + f\chi^2\sinh\chi\vartheta\right], \tag{47b}$$

$$\overset{3}{A}_{1i} = \overset{3}{A}_{1in}/\chi D_1, \quad \overset{3}{A}_{1in} = d\sin\vartheta + e\chi\left(\chi^2+1\right)\sin\vartheta\cosh\chi\vartheta + \mathcal{K}_\gamma\left[b + e\chi\left(\chi\sin\vartheta\sinh\chi\vartheta - \cos\vartheta\cosh\chi\vartheta\right) - d\cos\vartheta\right], \tag{47c}$$

$$\overset{3}{A}_{2i} = \overset{3}{A}_{2in}/D_2, \quad \overset{3}{A}_{2in} = a\chi\left(\chi^2+1\right)\cosh\chi\vartheta + c\chi\left(\chi^2+1\right)\cos\vartheta + f\chi\left(\chi^2+1\right)\cos\vartheta\cosh\chi\vartheta +$$
$$+\,\mathcal{K}_\gamma\left[a\left(\chi^2+1\right)\sinh\chi\vartheta + c\chi\left(\chi^2+1\right)\sin\vartheta + f\chi\left(\chi\cos\vartheta\sinh\chi\vartheta + \sin\vartheta\cosh\chi\vartheta\right)\right], \tag{47d}$$

$$\overset{4}{A}_{1i} = \overset{4}{A}_{1in}/D_2, \quad \overset{4}{A}_{1in} = -a - c\chi^3\sin\vartheta\sinh\chi\vartheta - c\cos\vartheta\cosh\chi\vartheta + cM\vartheta\chi\left(\chi^2+1\right)\cos\vartheta\sinh\chi\vartheta - f\cos\vartheta +$$
$$\mathcal{K}_\gamma\left[-aM\vartheta - c\left(\chi^2+1\right)\sin\vartheta\cosh\chi\vartheta + cM\vartheta\chi\left(\chi\cos\vartheta\cosh\chi\vartheta + \sin\vartheta\sinh\chi\vartheta\right) - f\sin\vartheta\right], \tag{47e}$$

$$\overset{4}{A}_{2i} = \overset{4}{A}_{2in}/\chi D_1, \quad \overset{4}{A}_{2in} = -b\chi\left(\chi^2+1\right)\sinh\chi\vartheta + dM\vartheta\chi\left(\chi^2+1\right)\sin\vartheta\sinh\chi\vartheta + d\left(\chi^3\cos\vartheta\sinh\chi\vartheta - \sin\vartheta\cosh\chi\vartheta\right) - e\chi\left(\chi^2+1\right)\sin\vartheta +$$
$$+\,\mathcal{K}_\gamma\left[-b\left(\chi^2+1\right)\cosh\chi\vartheta + d\left(\chi^2+1\right)\cos\vartheta\cosh\chi\vartheta + dM\vartheta\chi\left(\chi\sin\vartheta\cosh\chi\vartheta - \cos\vartheta\sinh\chi\vartheta\right) + e\chi\left(\chi^2+1\right)\cos\vartheta\right]. \tag{47f}$$

are the solutions sought.

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