# ACCURACY OF FREQUENCY EVALUATION PERFORMED WITH THE PySINC SOFTWARE

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***Abstract:*** *In this paper we present a direct method to determine the harmonic components of a signal. The method bases on the fact that the distribution of the samples in the spectrum derived with the Discrete Fourier Transform (DFT) follow the sinc function. We found a correction term that places the estimated frequency on an inter-line position. Therefore, the method implies first to find the two DFT samples that belong to the main lobe and then to simply calculate the correction term and add it to the frequency of the sample found on the left side of the main lobe. We created the PySINC software that uses this method, which is written in the Python language. Tests performed to find the accuracy of the frequency estimates show that the results are very accurate and we concluded that this method can be used with confidence for modal analysis issues.*
***Keywords:*** *frequency evaluation, Discrete Fourier Transform, sinc function, spectral line, modal analysis*

## 1. INTRODUCTION

For numerous engineering problems estimating the natural frequencies of structures accurately is essential. The Discrete Fourier Transform (DFT) and its derivate versions as the Power Spectrum (PS) or Power Spectral Density (PSD) are influenced by the acquisition time, so these are proved to be not sufficiently accurate for engineering applications [1-4]. Windowing decreases the noise introduced by spectral leakage but does not improve the estimate. Some techniques that imply signal post-processing can be used to improve the estimation results. The most simple make use of interpolation involving two or three DFT samples [5-12]. The maximizer and its largest neighbor, or the two neighbors in other cases, are used. Another approach is zero-padding the original signal by adding to it points with zero amplitudes. In the frequency domain, the spectral lines become denser and the estimation error is diminished [13]. But this implies a cost, the amplitude decreases proportionally with the signal lengthening. An alternative different method to precisely identify the frequencies is to generate three spectra by iteratively cropping the signal and using maximizer for interpolation [14-16]. In this way, the dependence on the acquisition time is practically eliminated. The disadvantage of this method consists in the asymmetric evolution of the peaks in the overlapped spectrum, the maximizer being distributed after a pseudo-sinc function [17].
In prior research, we studied the leakage phenomenon and considered it to improve the frequency estimation [18]. The present paper is dedicated to present a method based on particularities of the DFT and the *sinc* function that permits obtaining an accurate frequency estimate.

## 2. THE PROPOSED ESTIMATOR

The representation of a harmonic signal $f(t) = A\cos(2\pi ft)$ with amplitude $A$ and frequency $f$ has the discrete representation given in:

$$\{x_k\} = \{x[0], x[1], ..., x[k], ..., x[N-1]\} \tag{1}$$

In equation (1), $x[k]$ is the value acquired at the $k$-th measurement and $N$ is the total number of samples used to describe the signal of time length $t_S$. The sampling resolution is calculated as $\Delta t = t_S/(N-1)$ .The signal is represented in the frequency domain at spectral lines which are multiples of the frequency resolution $\Delta f = 1/t_S$. The multiplication factor is the actually the spectral line number $j = 0...N-1$. If the frequency $f$ of the signal is not a multiple of $\Delta f$ , the harmonic component is misrepresented. The amplitude displayed at the spectral line that is closest to the true frequency $f$ has the greatest amplitude, but several amplitudes are also displayed at other spectral lines. This is referred to as spectral leakage in the literature. However, the sum of amplitudes of the Fourier series components is equal with the amplitude of the signal in the time domain.

The Fourier series representation of the harmonic signal is given by:

$$f(t) = \sum_{j=1}^{N-1} a_j \cos(2\pi j\Delta f t) + \sum_{j=1}^{N-1} b_j \sin(2\pi j\Delta f t) \tag{2}$$

The real coefficient $a_j$ and the imaginary coefficient $b_j$ are, for the signal having the time length $t_S$, given by:

$$a_j = \frac{2A}{t_s}\int_0^{t_s}\cos(2\pi f t)\cos(2\pi j\Delta f t)dt \quad \text{and} \quad b_j = \frac{2A}{t_s}\int_0^{t_s}\cos(2\pi f_e t)\sin(2\pi j\Delta f t)dt \tag{3}$$

After solving the integrals and neglecting the terms having at the denominator $(f + j\Delta f)$, we obtain:

$$a_j = A\frac{\sin[2\pi(f - j\Delta f)t_S]}{2\pi(f - j\Delta f)t_S} \quad \text{and} \quad b_j = A\frac{1-\cos[2\pi(f - j\Delta f)t_S]}{2\pi(f - j\Delta f)t_S} \tag{4}$$

The amplitude displayed at the $j$-th spectral line is found as:

$$X_j = A\sqrt{\left(a_j^2 + b_j^2\right)} \tag{5}$$

It was shown that

$$a_j^2 + b_j^2 = \frac{\left\{\sin[2\pi(f - j\Delta f)t_S]\right\}^2 + \left\{1-\cos[2\pi(f - j\Delta f)t_S]\right\}^2}{\left[2\pi(f - j\Delta f)t_S\right]^2} \tag{6}$$

and, after performing calculations, we obtain:

$$a_j^2 + b_j^2 = \frac{1+\left\{\sin[2\pi(f - j\Delta f)t_S]\right\}^2 + \left\{\cos[2\pi(f - j\Delta f)t_S]\right\}^2 - 2\cos[2\pi(f - j\Delta f)t_S]}{\left[2\pi(f - j\Delta f)t_S\right]^2} \tag{7}$$

Because the sum of the squared *sine* and *cosine* functions is the unit, equation (7) results in:

$$a_j^2 + b_j^2 = 2\frac{1-\cos[2\pi(f - j\Delta f)t_S]}{\left[2\pi(f - j\Delta f)t_S\right]^2} \tag{8}$$

We know that

$$\cos[2\pi(f - j\Delta f)t_S] = \cos[\pi(f - j\Delta f)t_S + \pi(f - j\Delta f)t_S] = \cos^2[\pi(f - j\Delta f)t_S] - \sin^2[\pi(f - j\Delta f)t_S] \tag{9}$$

Substituting equation (9) in equation (8) we obtain:

$$a_j^2 + b_j^2 = 2\frac{1-\cos^2[\pi(f - j\Delta f)t_S] + \sin^2[\pi(f - j\Delta f)t_S]}{\left[2\pi(f - j\Delta f)t_S\right]^2} = 2\frac{2\sin^2[\pi(f - j\Delta f)t_S]}{\left[2\pi(f - j\Delta f)t_S\right]^2} \tag{10}$$

Finally, substituting equation (10) in equation (5) results:

$$X_j = A\sqrt{\frac{4\sin^2[\pi(f - j\Delta f)t_S]}{\left[2\pi(f - j\Delta f)t_S\right]^2}} = A\frac{\sin[\pi(f - j\Delta f)t_S]}{\pi(f - j\Delta f)t_S} = A\mathrm{sinc}[\pi(f - j\Delta f)t_S] \tag{11}$$

If $f = j\Delta f$, the *sinc* function takes the value one and the real amplitude $A$ is indicated at the $j$-th spectral line as $X_j$. Else, $X_j < A$ and in consequence neither the frequency nor the amplitude are correctly found. For the latter case, we represent the *sinc* function and the two spectral lines that belong to the main lobe in Figure 1.

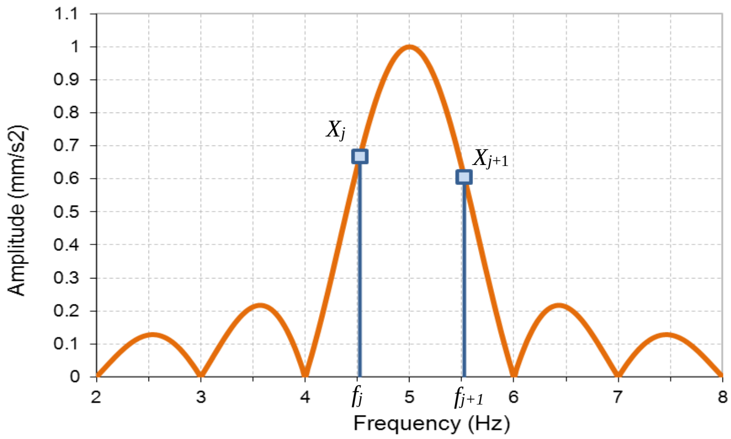


**Figure 1:** The spectrum showing the DFT samples on the main lobe and the *sinc* function

We know that, when $f - j\Delta f > 0$, the spectral line $j$ is at the left side of the main lobe. Let now $X_j$ and $X_{j+1}$ be the amplitudes found at spectral lines $j$ and $j$+1. To find the true frequency $f_e$ we have to solve the system:

$$\begin{cases} X_j = \dfrac{\sin[\pi(f_e - j\Delta f)t_S]}{\pi(f_e - j\Delta f)t_S} \\ X_{j+1} = \dfrac{\sin\{\pi[f_e - (j\Delta f + \Delta f)]t_S\}}{\pi[(f_e - (j\Delta f + \Delta f)]t_S} \end{cases} \tag{12}$$

From the second equation in (12), we have that:

$$X_{j+1} = \frac{\sin\{\pi[(f_e - j\Delta f) - \Delta f]t_S\}}{\pi[(f_e - j\Delta f) - \Delta f]t_S} = \frac{\sin\left[\pi(f_e - j\Delta f)t_S - \pi\Delta f \cdot t_S\right]}{\pi(f_e - j\Delta f)t_S - \pi\Delta f \cdot t_S} \tag{13}$$

and knowing that $\Delta f \cdot t_S = 1$ and $\sin(x - \pi) = -\sin(x)$, equation (14) becomes

$$X_{j+1} = \frac{\sin\{\pi[(f_e - j\Delta f) - \Delta f]t_S\}}{\pi[(f_e - j\Delta f) - \Delta f]t_S} = \frac{\sin\left[\pi(f_e - j\Delta f)t_S - \pi\Delta f \cdot t_S\right]}{\pi(f_e - j\Delta f)t_S - \pi\Delta f \cdot t_S} = \frac{-\sin\left[\pi(f_e - j\Delta f)t_S\right]}{\pi\left[(f_e - j\Delta f)t_S - 1\right]} \tag{14}$$

Hence, the system of equations (12) becomes

$$\begin{cases} X_j = \dfrac{\sin\left[\pi(f_e - j\Delta f)t_S\right]}{\pi(f_e - j\Delta f)t_S} \\ X_{j+1} = -\dfrac{\sin\left[\pi(f_e - j\Delta f)t_S\right]}{\pi\left[(f_e - j\Delta f)t_S - 1\right]} \end{cases} \tag{15}$$

After simple mathematical handling results

$$X_j\left[(f_e - j\Delta f)t_S\right] = -X_{j+1}\left[(f_e - j\Delta f)t_S - 1\right] \tag{16}$$

which can be grouped as

$$t_S X_{j+1} f_e + t_S X_j f_e = t_S X_{j+1} j\Delta f + t_S X_j j\Delta f + X_{j+1} \tag{17}$$

resulting the expression of the estimated frequency

$$f_e\left(X_{j+1} + X_j\right)t_S = j\Delta f\left(X_{j+1} + X_j\right)t_S + X_{j+1} \tag{18}$$

Hence, the estimated frequency is found from the mathematical relation:

$$f_e = j\Delta f + \frac{X_{j+1}}{\left(X_{j+1} + X_j\right)t_S} = j\Delta f + \delta \tag{19}$$

where

$$\delta = \frac{X_{j+1}}{\left(X_{j+1} + X_j\right)t_S} \tag{20}$$

$\delta$ is a correction term that helps finding the true frequency at an inter-line position. Thus, the frequency can be estimated when we know the values $X_j$ and $X_{j+1}$ displayed at lines $j$ and $j$+1 belonging to the main lobe. It does not matter which is the bigger component, always that found at the left lobe side is considered as a reference.

## 3. THE DEVELOPED SOFTWARE

The software developed for frequency estimation, nominated PySINC because it involves a sinc-based algorithm, performs following steps:

1. Imports the original signal from CSV or LVM files
2. Asks for the maximum targeted frequency
3. Add two samples with zero amplitude to the original signal and obtain a zero-padded signal
3. Calculates the DFT of the zero-padded signal and display it (Figure 1, upper diagram)
4. Calculates the DFT of the original signal
5. Asks for the threshold amplitude to know the peaks for which it continues processing
6. For each individual peak it determines the evolution of the maximizer considering the two DFTs
7. Find the spectral lines located on the main lobes in function of the evolution of the maximizer
8. Calculate $\delta$ for the DFT of the original and the true frequency with equations (20) and (19)
9 Displays the DFT of the original signal including the sinc function (Figure 1, bottom diagram)
10. Displays the estimated frequency and amplitude in a pop-up window.

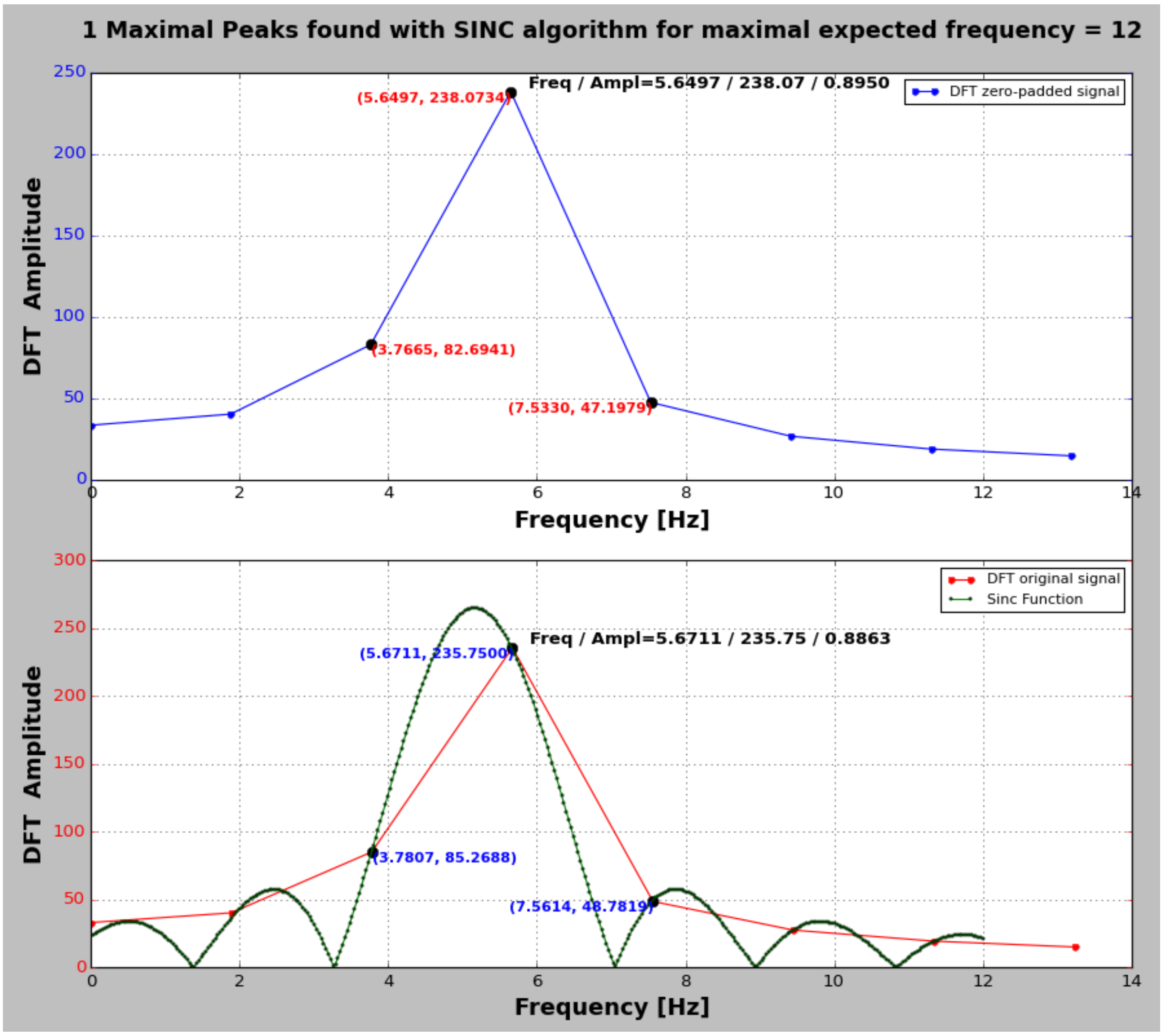


**Figure 2:** The interface of the Frequencies obtained with the standard DFT and the developed PySINC software

Handling the software is quite easy, it is just necessary to introduce the value of the maximum targeted frequency (at position 2 in the workflow described at the start of this section) and indicate by right-clicking the threshold (position 5 in the workflow). The user can see in the second displayed diagram if the DFT samples of the main lobe and the *sinc* function plotted for the estimated frequency fit. If this is not the case, the estimation is incorrect. From our experience this happens just if the signal is undersampled, i.e. it contains fewer samples as recommended by the Nyquist sampling theorem.

## 4. RESULTS AND DISCUSSION

In order to proof the method and implicitly the PySINC software's reliability we performed numerous simulations for generated signals, hence with known frequency. In this section we present the results obtained for 2 and 3 cycles of a sinusoid with the frequency $f$ = 5.17 Hz and amplitude $A$ = 1 m/s$^2$. The signal was generated with a sampling rate $r$ = 1000 and the number of samples was $N$ = 400...700. The number of samples was increased with the step $s$ = 20 but in particular regions it was denser to obtain precise curves. We calculated the maximum theoretically achievable error as $\varepsilon = \Delta f/2$, the frequency with the standard DFT and the frequency with the PySINC software which involves the *sinc*-based algorithm. We plot in Figure 3 the frequency curves. One can observe the high accuracy obtained with PySINC dissimilar with the raw frequency estimation achieved with the standard DFT.

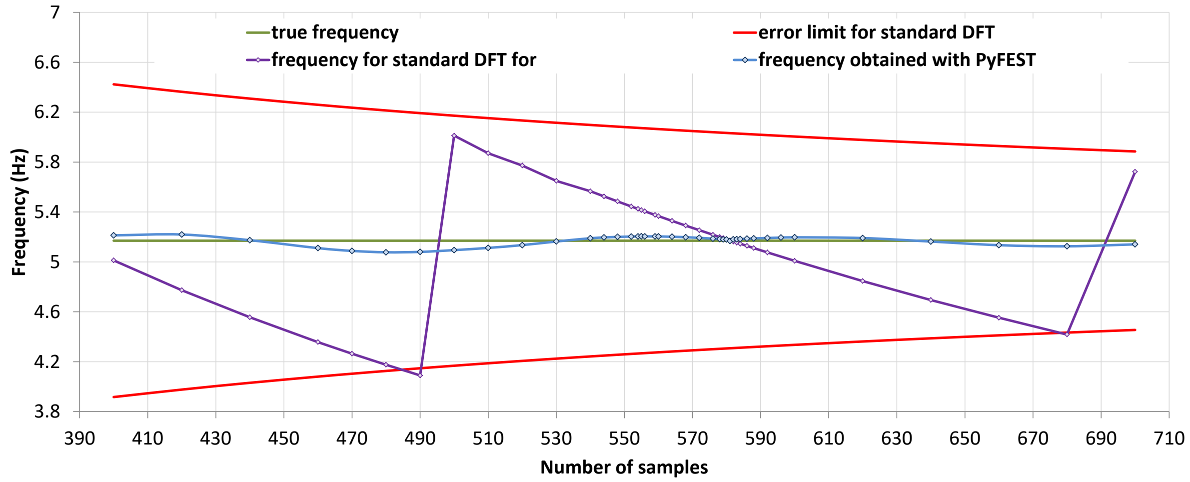


**Figure 3:** Frequencies obtained with the standard DFT and the developed PySINC software

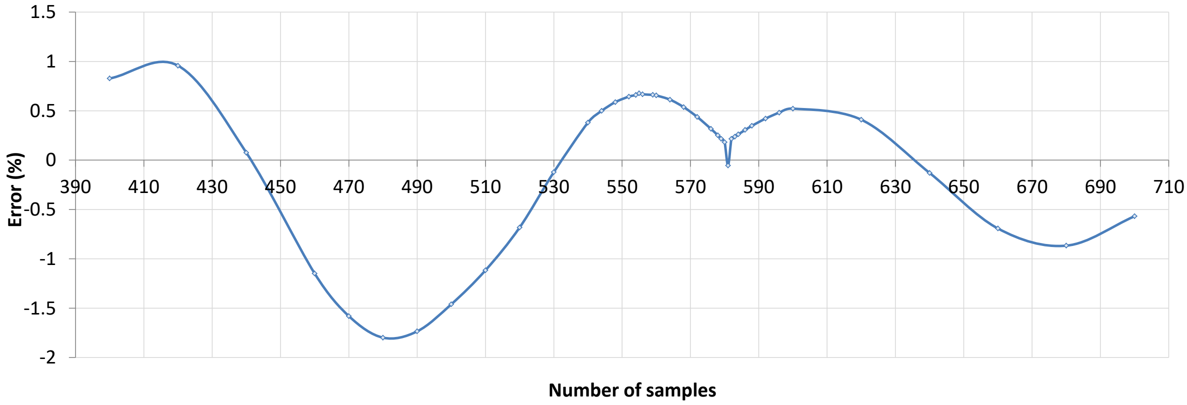


**Figure 4:** Errors in frequencies estimation resulted when using the developed PySINC software

We also calculate the error achieved when involving the sinc-based method and show the small resulted errors in Figure 4. These are, obviously, much smaller as those obtained with the standard DFT. This makes us considering that the estimation is sufficient precise to permit a fine evaluation of engineering processes, as damage detection is. Tests made for incipient damage were successfully performed [19-20], the occurrence of cracks being possible in the very early stage.

## 5. CONCLUSION

The paper introduces a sinc-based algorithm to estimate the harmonic components of a signal with high accuracy. The algorithm is implemented in software, written in the Python language, which has been proved to be reliable even for signals generated with a short time length. The software is easy to be used and provide accurate results. For the tests made involving a signal with the known frequency of 5.17 Hz, we obtained errors less than 2% even for a very low number of cycles. Obviously, by increasing a little bit the time length of analysis, the error is dramatically reduced. However, also the errors achieved for a short signal are reasonable for most engineering applications and recommend the software for practical use.

## REFERENCES


[1] Ostachowicz W.M., Krawczuk C., Analysis of the effect of cracks on the natural frequencies of a cantilever beam, Journal of Sound and Vibration, 150(2), 1991, pp. 191–201.

[2] Vlase S., Marin M., Öchsner A., Considerations of the transverse vibration of a mechanical system with two identical bars, Proceedings of the Institution of Mechanical Engineers, Part L: Journal of Materials: Design and Applications, 233(7), 2019, pp. 1318-1323.

[3] Tufoi M., Hatiegan C., Vasile O., Gillich G.R., Dynamic Analysis of Thin Plates with Defects by Experimental and FEM Methods, Romanian Journal of Acoustics and Vibration,10(2), 2013, pp. 83-88.
[4] Bratu P., Elastic systems vibrations, Technical Publishing House, Bucharest, 2000.
[5] Rife D.C., Vincent G.A., Use of the discrete Fourier transform in the measurement of frequencies and levels of tones, Bell Systems Technology Journal, 49(2), 1970, pp. 197-228.
[6] Grandke T., Interpolation Algorithms for Discrete Fourier Transforms of Weighted Signals, IEEE Transactions on Instrumentation and Measurement, Vol. 32, 1983, pp. 350-355.
[7] Quinn B.G., Estimating Frequency by Interpolation Using Fourier Coefficients, IEEE Transactions on Signal Processing, Vol. 42, 1994, pp. 1264-1268.
[8] Jain V.K., Collins W.L., Davis D.C., High-Accuracy Analog Measurements via Interpolated FFT, IEEE Transactions on Instrumentation and Measurement, Vol. 28, 1979, pp. 113-122.
[9] Ding K., Zheng C., Yang Z., Frequency Estimation Accuracy Analysis and Improvement of Energy Barycenter Correction Method for Discrete Spectrum, Journal of Mechanical Engineering, 46(5), 2010, pp. 43-48.
[10] Voglewede P., Parabola approximation for peak determination, Global DSP Magazine, 3(5), 2004, pp. 13-17.
[11] Jacobsen E., Kootsookos P., Fast, accurate frequency estimators, IEEE Signal Processing Magazine, 24(3), 2007, pp. 123-125.
[12] Yao S., Wu Q., Fang S., An Improved Fine-Resolution Method for Frequency Estimation of Real-Valued Single-Tone Using Three DFT Samples, IEEE Access, 7, 2019, pp. 117063-117074.
[13] Gillich G.R., Mituletu I.C., Signal Post-processing for Accurate Evaluation of the Natural Frequencies. In: R. Yan, X. Chen, S. Mukhopadhyay (eds), Structural Health Monitoring. Smart Sensors, Measurement and Instrumentation, Vol. 26, Springer Cham, 2017, pp. 13-37.
[14] Ntakpe J.L., Gillich G.R., Mituletu I.C., Praisach Z.I., Gillich N., An Accurate Frequency Estimation Algorithm with Application in Modal Analysis, Romanian Journal of Acoustics and Vibration, 13(2), 2016, pp. 98-103.
[15] Gillich G.R., Mituletu I.C., Negru I., Tufoi M., Iancu V., Muntean F., A Method to Enhance Frequency Readability for Early Damage Detection, Journal of Vibration Engineering & Technologies, 3(5), 2015, pp. 637-652.
[16] Minda P.F., Praisach Z.I., Gillich N., Minda A.A., Gillich G.R., On the Efficiency of Different Dissimilarity Estimators Used in Damage Detection, Romanian Journal of Acoustics and Vibration, 10(1), 2013, pp. 15-18.
[17] Gillich G.R., Mituletu I.C., Praisach Z.I., Negru I., Tufoi M., Method to Enhance the Frequency Readability for Detecting Incipient Structural Damage, Iranian Journal of Science and Technology, Transactions of Mechanical Engineering, 41 (3), 2017, pp. 233–242.
[18] Minda A.A., Gillich G.R., Sinc Function based Interpolation Method to Accurate Evaluate the Natural Frequencies, Analele Universitatii Eftimie Murgu Resita, 24(1), 2017, pp. 211-218.
[19] Gillich G.R., Maia N., Mituletu I.C., Tufoi M., Iancu V., Korka Z., A new approach for severity estimation of transversal cracks in multi-layered beams, Latin American Journal of Solids and Structures, 13(8), 2016, pp. 1532-1544.
[20] Gillich G.R., Furdui H., Wahab M.A., Korka Z.I., A robust damage detection method based on multi-modal analysis in variable temperature conditions, Mechanical Systems and Signal Processing, 115, 2019, pp. 361-379.