SOLUTIONS AND APLICATIONS REGARDING DYNAMICAL SYSTEMS ABSOLUTE STABILIZATION AND OPTIMAL CONTROL

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Abstract: *The first part of the paper deals with criteria and methods to control systems absolute stabilization. The second part focuses on variational methods and systems optimal control methods: Lurie, Euler-Lagrange and Pontreaguine principles to obtain minimal time or optimal fuel or energy consumption.*
The examples included in the paper are from mechanics, electromagnetism, hydro-aero-dynamics, etc.
Keywords: *Absolute stabilization, variational methods, optimal control*

1. INTRODUCTION

The automatic regulation of dynamical systems stability occupies a fundamental position in nowadays science and technique, following the optimization of the technological process of the cutting tools, of the robots, of the movement vehicles regime or of some machines components, of energetic radioactive regimes, chemical, electromagnetic, thermal, hydro-aerodynamic regimes, etc. The studies and the technical achievements are complex, involving mathematical models for close-loop circuits, following for the automatic regulation the integration of some mechanisms and devices with inverse reaction of response for the control and the fast and efficient elimination of the perturbations which can appears along these processes or dynamical regimes. Generally, these dynamical regimes are nonlinear and it was necessarily some contributions and special achievements for automatic regulation, generating the automatic regulation of absolute stability (a.r.a.s.) for these classes of nonlinearities.
We highlight two special methods (a.r.a.s.):

- Liapunov's function method discovered by A.I. Lurie [5] and developed into a series of studies by M.A Aizerman, V.A. Iakubovici, F.R. Gantmaher, R.E. Kalman, D.R. Merkin [6] and others.
- Frequency method developed by researcher VM Popov [6] generalizing the criterion of Nyquist, then developed in many further studies.

We note the contributions of Romanian researchers recognized by the works and monographs on the stability and optimal control theory: C. Corduneanu, A. Halanay, V. Barbu, Th. Morozan, G. Dinca, M. Megan, Vl. Rasvan, V. Ionescu, M.E. Popescu, S. Chiriacescu, A. Georgescu and also who studied directly on (a.r.a.s.): I. Dumitrache, D. Popescu, C. Belea, V. Rasvan, S. Chiriacescu and other recent works.
The theorem Kalman -Yakubovich - Popov (K-Y-P) highlights the fact that the two methods are equivalent. The study in the case of automatic regulation for the absolute stabilization of the crafts dynamics in the case of rolling perturbations will be made. After obtaining of the stability the optimal control will be made with the Pontreaguine extremum principle using the criterion of minimal time of the stabilization and the rolling absorption.
It will be considered the rectilinear and uniform course of an aircraft in horizontal plane. We take *Oxyz* the axis system (*xOy* - horizontal and *xOz* - vertical). The horizontal axis $x'Ox$ is parallel with the longitudinal axis of the craft that pass through the centroid G. During the flight some perturbations could appear due to some external ot internal forces. Thus, it will be determined the momentum that is highlighted by the rotation angles around these axis (figure 1):

- $\angle\psi$ is the rotation angle around *Oy* axis - pitch angle
- $\angle\theta$ the rotation angle around *Ox* axis - rolling angle
- $\angle\Phi$ the rotation angle around *Oz* axis - angle of yaw

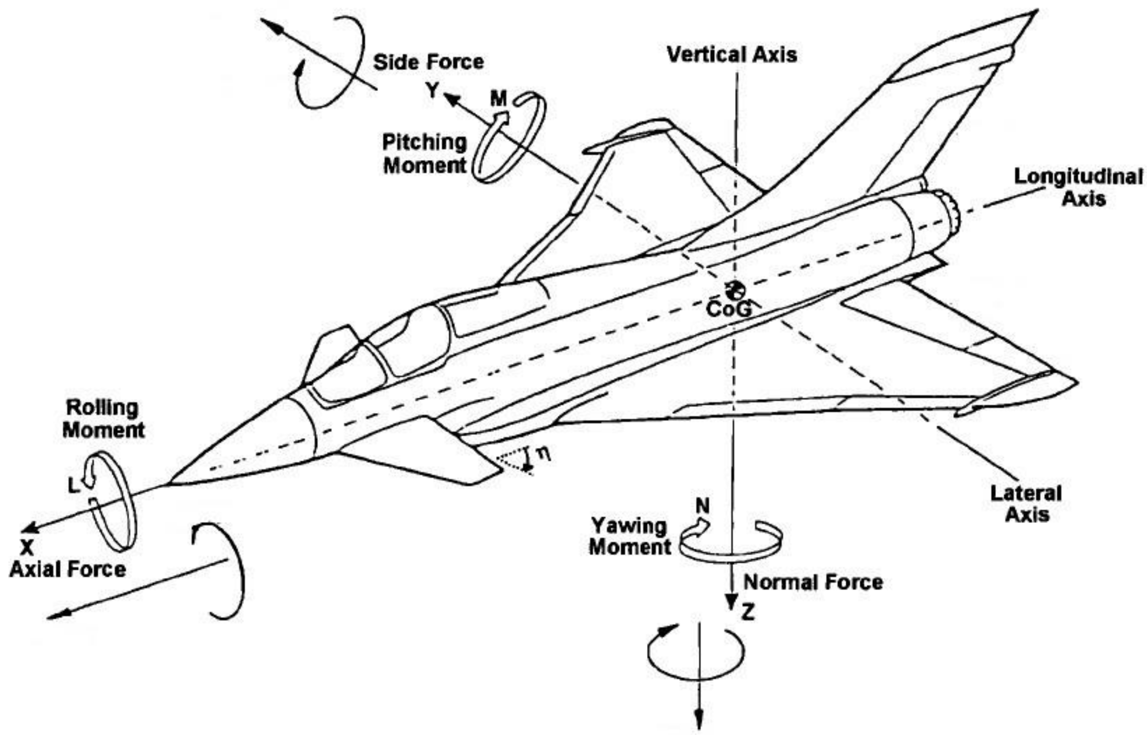

Figure 1: The momentum in the case of aircraft dynamics

Some disconnected studies regarding the longitudinal stability (pitching) and the lateral stability (rolling and gyration) will be presented in the following chapters of this paper. The movement equations related to (ψ, θ, Φ) form a nonlinear system of 6^{th} degree. Separately, it was made the disconnected study regarding the longitudinal stability of the flight, namely the pitching problem with the automatic regulation (a.r.a.s.) and autopilot.

The lateral perturbations can be:

• atmospheric, air gaps, wind, waves which determine the gyration and that implies rolling (and reciprocally)

• faults on speed - frequency (stop) the propelling engine (2-4-6 traction engine) that produce swerving combining rolling with gyration.

• failure of the landing gear

• very sensitive are the supersonic aircraft (with high speed) with the wings in V - the fuselage is located on the bottom generation a dihedral effect.

The stability regulation is made with ailerons, shutters, flaps, fluid shutters with jet (blowing or absorption). These conjugated turnings are directed by an automatic regulator with levers, mechanisms, hydro-pneumatic systems (anti-swing). It is equipped with sensors, measurement and control devices, gyroscopes with rigid quickly response or controllers for the regulation of the engine's speed. These automatic regulations are encountered at crafts, rockets, torpedoes, crafts with vertical take-off and landing.

The study of the stability and the automatic stabilization in rolling case with the variables: θ – rolling angle and $\dot{\theta}$ – (angular) speed.

The horizontal tail assembly controls the rolling, and the direction is controlled by the rudder airplane that highlights the gyration.

The side streams can perturb the direction producing gyration and sideslip. In such case the symmetry of the aerodynamic form is modified and the bearing wings varies and implies the rolling. The swing depends on the yaw momentum.

2. ARAS AND OPTIMAL CONTROL IN CASE OF ROLLING PERTURBATIONS

As mentioned before, it will be considered the longitudinal flight of the aircraft or missile - uniform, without lateral disturbances, asymmetries or gyrations; we only have rolling angle $\theta(t)$ perturbations. Suppose a perturbed trajectory that starts at the moment $t_0 = 0$ from $x^0(0) = \alpha$ and it may be stabilized in minimal time t^* in the origin O^*.

The optimal control u^* and minimum transfer time t^* will be determined. The linearized equation of the roll angle $\theta(t)$ with respect to the horizontal axis Ox is:

$$T\ddot{\theta} + \dot{\theta} = -k \quad (1)$$

where *T* is the time constant, *k* is a constant (i.e. the product of the coefficients of rolling and gyration moments) and *u(t)* is the control function.

In the phase plane (x_1, x_2) with $\theta = x_1$ we have:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{T}x_2 - \frac{k}{T}u \end{cases}; \quad k > 0, T > 0 \quad (2)$$

The system (2) is linear and autonomous with:

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, A = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{1}{T} \end{pmatrix}, B = \begin{pmatrix} 0 \\ -\frac{k}{T} \end{pmatrix} \quad (3)$$

KALMAN: The system is controllable if $rang(B, AB) = 2$.

It is also absolutely stabilizable in origin, with its Eigen values $r_1 = 0, r_2 = -\frac{1}{T}$.

At the initial moment the perturbed system was in position:

$$x_1(0) = \alpha_1, x_2(0) = \alpha_2 \quad (4)$$

and we want it to be brought to the final (stable) position O(0,0) in minimum time, t^*, that is $x(t^*) = 0$:

$$x_1(t^*) = 0, x_2(t^*) = 0 \quad (5)$$

Note that the states $x(0) = \alpha$, $x(t^*) = 0$ and $t_0 = 0$ are fixed, and the final time t_1 has a free horizon and must be determined $\min(t_1) = t^*$, with $I[u^*] = \min(t_1 - t_0) = t^*$.

Here the order of the system is $n = 2$ and for the parameter $u(t) = u_1, m = 1$.

The allowable domain for $u(t)$ is:

$$U = \{u | u \in [-1, 1]\} \quad (6)$$

We apply the principle of the minimum, the goal (criterion) being to minimize the stabilization time. In order to simplify the analysis of the optimal control and the study of the solutions we will make a non-singular linear transformation of the system, $x \leftrightarrow z$:

$$x = Pz, \dot{x} = P\dot{z};\ z = P^{-1}x; (x_1, x_2) \leftrightarrow (z_1, z_2),\ A \leftrightarrow M \quad (7)$$

We will determine the matrix $P_{2\times2}$ so that $M = \begin{pmatrix} 0 & 0 \\ r_1 & r_2 \end{pmatrix}$. With substitutions (7), from (2) – (5) it results:

$$\dot{z} = Mz + P^{-1}Bu, \begin{cases} \dot{z}_1 = -ku \\ \dot{z}_2 = -\frac{1}{T}z_2 + ku \end{cases} \quad (8)$$

$$M = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{T} \end{pmatrix};\ P = \begin{pmatrix} 1 & 1 \\ 1 & -\frac{1}{T} \end{pmatrix},\ detP = -1 - \frac{1}{T} \neq 0 \quad (9)$$

Relationships will be transformed and bilocal conditions will be:

$$\begin{cases} x_1 = z_1 + z_2 \\ x_2 = -\frac{z_1}{T} \end{cases};\ \begin{cases} z_1 = -Tx_2 \\ z_2 = x_1 + Tx_2 \end{cases} \quad (10)$$

$$z(0) = \alpha;\ z_1^0 = \alpha_1;\ z_2^0 = \alpha_2 z(t_1) = 0;\ z_1(t^*) = 0,\ z_2(t^*) = 0 \quad (11)$$

This non-singular transform preserves the dynamic and stable states:

$$(x_1^0, x_2^0) \leftrightarrow (\alpha_1, \alpha_2);\ \left(x_1^1(t^*), x_2^1(t^*)\right) \leftrightarrow (0,0).$$

We build the Hamiltonian $H[t, z, u, \lambda]$:

$$H = 1 + \lambda_1\dot{z}_1 + \lambda_2\dot{z}_2,\ t \in [0, t_1] \quad (12)$$

$$\dot{z} = \frac{\partial H}{\partial \lambda};\ \dot{\lambda} = -\frac{\partial H}{\partial \dot{z}} \quad (13)$$

$$\dot{\lambda} = -M'\lambda,\ \lambda = \lambda(t) \rightarrow \lambda_1(t), \lambda_2(t) \quad (14)$$

and M' is the transpose of matrix *M*.

We make these systems more explicit related to H:

$$H[t, z, u, \lambda] = 1 - \frac{\lambda_2}{T}z_2 + u(t)[\lambda_2(t) - \lambda_1(t)]k \quad (15)$$

In (15) it may be noticed that $H = H[u]$ is continuous for $u \in [-1,1]$ and it's shape is linear, so the extremes of *H[u]* will be at the ends of this interval:

$$u^* = \pm1,\ H_{min} = H[u^*, t^*] = H^* \quad (16)$$

for

$$u^* = -sgn(\lambda_2 - \lambda_1),\ t \in [0, t^*] \quad (17)$$

In this case, with u^* from (17), we will determine H^*, t^* and the optimal trajectories of the systems (13), (14):

$$\begin{cases} \dot{z}_1^* = \frac{\partial H^*}{\partial \lambda_1} = -ku^* \\ \dot{z}_2^* = \frac{\partial H^*}{\partial \lambda_2} = -\frac{z_2^*}{T} + ku^* \end{cases} \quad (18)$$

$$\begin{cases} \dot{\lambda}_1^* = \frac{\partial H^*}{\partial z_1} = 0 \\ \dot{\lambda}_2^* = \frac{\partial H^*}{\partial z_2} = \frac{\lambda_2^*}{T} \end{cases} \quad (19)$$

We notice that (14) is verified by (19), $\lambda^*(0) = C$, so with the solutions of (18), (19) and taking into consideration the condition:

$$z_0^* = \alpha, z_1^*(t^*) = 0, \quad (20)$$

it results:

$$\begin{cases} z_1^*(t) = -ku^*t + \alpha_1 \\ z_2^*(t) = (\alpha_2 - ku^*T)e^{-\frac{1}{T}} + ku^*T \end{cases}, t \in [0, t_1] \quad (21)$$

$$\begin{cases} \lambda_1(t) = C_1 \\ \lambda_2(t) = C_2 e^{\frac{t}{T}} \end{cases} \quad (22)$$

$$u^* = sgn\left(C_1 - C_2 e^{\frac{t}{T}}\right), \ t \in [0, t_1] \quad (23)$$

FELDBAUM Theorem states that a n^{th}-order system can make at most (n-1) switches on n intervals.
From the switching equation (23) we notice that the sign can be changed at most once, i.e. when $\lambda_1 = \lambda_2$ (the root of the switching function: $F \equiv C_1 - C_2 e^{\frac{t}{T}} = 0$).

$$\tau^* = Tln\left(\frac{C_1}{C_2}\right), \ C_2 \neq 0, \ \frac{C_1}{C_2} > 0 \quad (24)$$

And for $0 < \tau^*$ we have $\frac{C_1}{C_2} \geq 1$.

As $\frac{C_1}{C_2} < 1$ case isn't possible, so there will be no relay switching for $t \in [0, t_1]$ and the extreme command will be $u_1^* = -1, \ u_2^* = +1$.

$C_1 = C_2 = 0$ is impossible too, because $\lambda \neq 0$.

We'll analyze the two possible cases.

Case I: No (relay) commutation

$$C_2 = 0, \quad \lambda_1^* = C_1, \quad \lambda_2^* = 0, \quad H[u^*, t^*] = 1 - kC_1 sgnC_1 = 0, \quad C_1 = \pm\frac{1}{k},$$

$$u^* = sgn\lambda_1 = sgnC_1, \ C_1 = \pm\frac{1}{k} \quad (25)$$

It comes out that for $t \in [0, t_1]$ there is no relay switching and the extreme solutions will be:
$u^* = -1$ dacă $C_1 < 0$ și $u^* = +1$ dacă $C_1 > 0$,
and the optimal trajectories (21) will be for $u^* = \pm 1$.

By imposing the final condition (20) at the final point (target) $z(t = t^*) = 0$ in (21) we obtain that the trajectories will pass through origin while stabilizing, i.e.:

$$\alpha_1 = ku^*t^*, \quad \alpha_2 = ku^*T\left(1 - e^{\frac{t^*}{T}}\right) \quad (26)$$

$$\min t_1 = t^* = \frac{|\alpha_1|}{k}, \quad u^* = \pm 1 \quad (27)$$

Subtracting t* from (26), we obtain that in the plane (z_1, z_2) the conditions that (α_1, α_2) must respect so that the trajectories will meet the origin in the moment t* will be:

$$\alpha_2 = \pm kT\left(1 - e^{\pm\frac{\alpha_1}{kT}}\right) \quad (28)$$

i.e.

$$z_2 = \pm kT\left(1 - e^{\pm\frac{z_1}{kT}}\right) \quad (29)$$

The upper sign corresponds to the C^+ curve and lower to the C^- (switching curve), where $z_2 \in (-kT, kT)$ as in the figure 2.

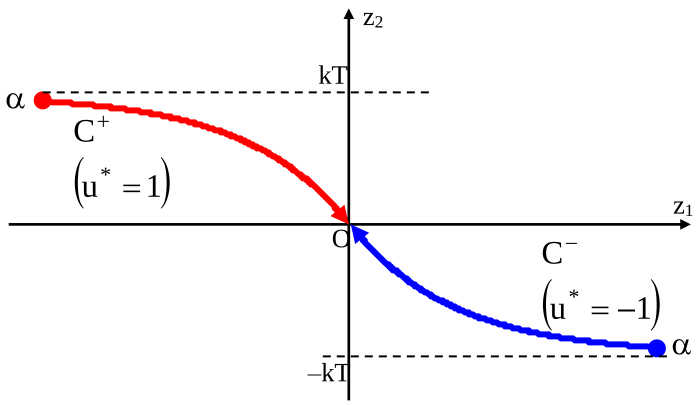

Figure 2: Un-commutated trajectories

Case II: General (with commutation)

$C_2 \neq 0$, $\frac{C_1}{C_2} > 0$: extreme control switches with (24) in the moment $\tau^* = Tln\left(\frac{C_1}{C_2}\right)$, $0 < \tau^* < t_1$. In this case the stabilization process will consist in two steps:

- the trajectories – of type $C'_{\pm}$ – will start at the moment $t_0 = 0$ from the point $\gamma(\gamma_1, \gamma_2)$ until the moment τ^* when $z^*(Q, \tau^*) = \beta$ and $u*$ switches and then for $t \in [\tau^*, t_1]$ the trajectories will change to those of type (29) starting at $t_0 = \tau^*$ from $Q(\beta)$, i.e. of type $C^{\pm}$ which reach origin $O(t_1 = t^*)$ obtaining $0 < \tau^* < t^*$. These trajectories are plotted in figure 3 and optimally we will have:

  $C^* = \{C'_+ \cup C^-\} \cup \{C'_- \cup C^+\}$. For $C'_{\pm}$ cu $z_1^*(t_0 = 0) = \gamma$ we obtain the solutions:

  $$z_1^* = \gamma_1 \pm kt,\ z_2^* = (\gamma_2 \pm kT)e^{-\frac{t}{T}} \mp kT,\ t \in [0, \tau^*] \tag{30}$$

  $$z_2^* - ku^*T = (\gamma_2 - kuT)e^{\frac{z_1^* - \gamma_1}{u^*kT}} \tag{31}$$

  $Q(\beta)$ coordinates are computed from (30) for $t = \tau^*$, $z_{1,2}^* = \beta_{1,2}$, i.e:

  $$\beta_1 = \gamma_1 - ku^*\tau^*,\quad \beta_2 = ku^*T + (\gamma_2 - ku^*T)e^{-\tau^*T} \tag{32}$$

- We pass to the next step, $t \in [\tau^*, t_1]$, where the trajectories start from $Q(\tau^*, t_1)$ and finally reach the origin $O(t_1 = t^*)$, i.e. $O(0,0)$; we notice that they are of type (29):

  $$t^* = \tau^* + \left|\frac{\beta_1}{ku^*}\right| = \frac{|\alpha_1|}{k} + \frac{|\beta_1|}{k}, \text{ according to FELDBAUM theorem.} \tag{33}$$

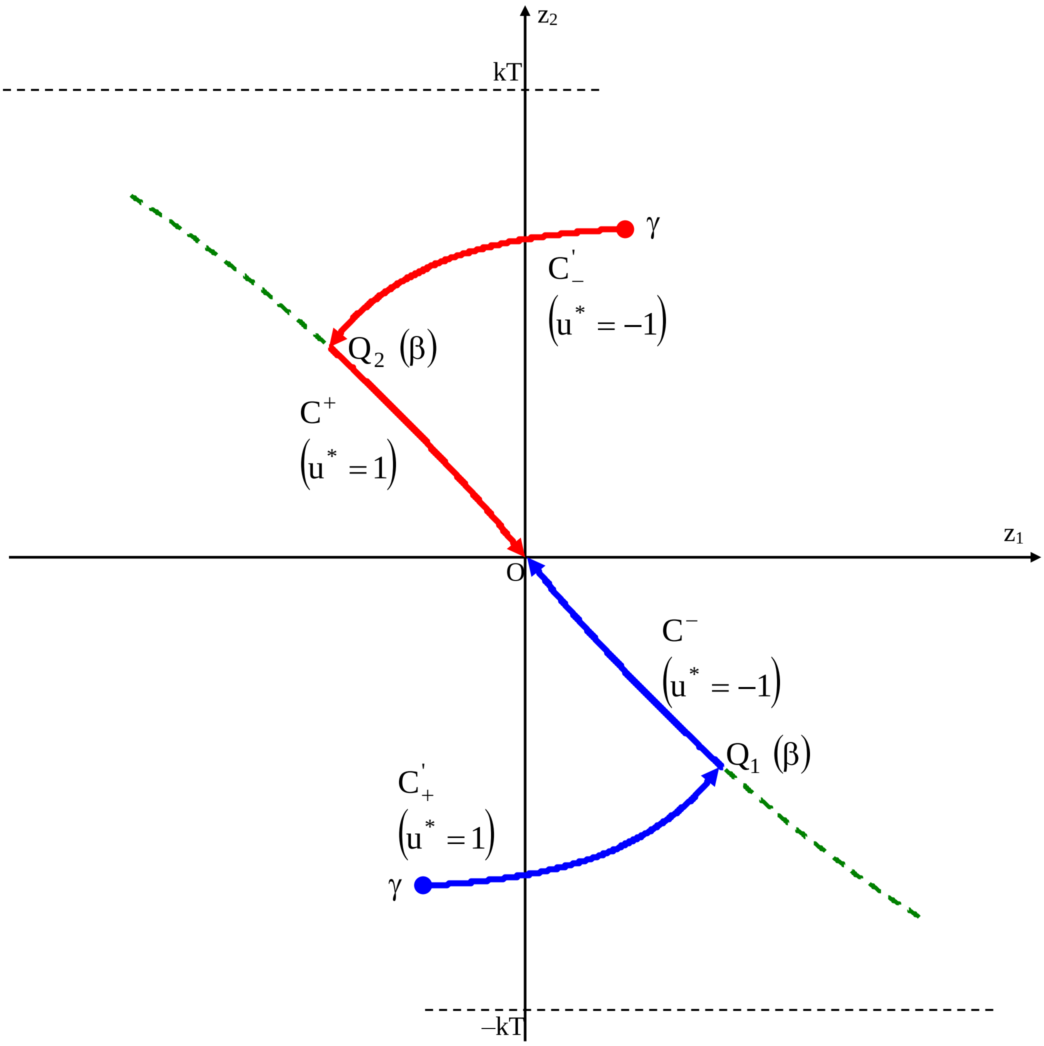

Figure 3: One-commutation trajectory

Substituting t^*, τ^*, u^*, r^* in $H = H^* = 0$ on the intervals $(0, \tau^*)$, (τ^*, β) gives the connection between C_1, C_2 – the parameters of the relay type regulator with switching point $t = \tau^*$.

The analysis and study are transmitted in the plane of the phases (x_1, x_2) with the relations (10), (11).

3. CONCLUSION

This paper focuses on a theoretical and applicative study of aircraft dynamics. Effectively, by using analytical graphical and numerical methods, a horizontally perturbed flying object is stabilized in minimal time. This study may be extended, taking into consideration more criteria for the optimal control, like a minimization of the fuel consumption, by using the same relay-type (on-off) regulator.
The microprocessor that controls the entire navigation process of the aircraft, including its (auto)stabilization, must be programmed with the results obtained in this work in order to correct the deviation from the course due to the roll perturbations.

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