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THE SPECTRAL RESPONSE OF SINGLE-DEGREE OF FREEDOM SYSTEMS WITH STATIONARY RESPONSE

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Abstract: *In many practical applications the system of concern has non-linearities, whitch must be taken into account if one is to predict its performance in a realistic way. For such non-linear problems the classical linear theory, which can be found in a number of text books, is not directly applicabile and new techniques are required. If a non-linear mathematical model of the system under consideration is adopted, together with a random process model of the excitation, then is faced with the problem of predicting the system response. A method for estimating the power spectral density of the stationary response of oscillator with a nonlinear restoring force subjected to external wide band noise excitation has been proposed. The efficiency of the method is checked by comparing results with those numerical simulations. The basic idea of the statistical linearization approach is to replace the original nonlinear system by a linear one. This is done in such a way that the difference between the two systems is minimised in some statistical sense. In this way, the parameters of the linearised system are determined.*
Keywords:*random vibration, equivalent linearization, spectral density function, power spectral density, response.*

1. SYSTEM MODEL

The research goals are, firstly, the computation of stochastic, nonlinear response characteristics (with accuracy and efficiency as important criteria) and, secondly, the investigation and thorough understanding of stochastic, nonlinear response phenomena. The desire to compute response characteristics, such as the power spectral density of the response of these systems, leads to the development of methods that can be used to approximate this response. The necessity of approximation is caused by absence of analytical solutions for general, nonlinear systems. The stochastic linearization technique can be considered to be an extension of the equivalent linearization method for the treatment of nonlinear systems under deterministic excitations. Much of the recent research effort in the field of random vibration has been directed towards developing suitable methods for analisyng non-linear systems. Methods of predicting the vibration response of mechanical and structural system to fluctuating external forces have grown rapidly in importance, in engineering design, over the last century.The basic idea of the statistical linearization approach is to replace the original nonlinear system by a linear one. This is done in such a way that the difference between the two systems is minimised in some statistical sense. In this way, the parameters of the linearised system are determined. The response of the nonlinear system is approximated by the response of the equivalent linear system. So, the unknown statistics of the response are evaluated approximating the response as a Gaussian process, when the excitation is assumed to be Gaussian.
To illustrate the procedure of equivalent linearization theory, let us consider the following oscillator with a nonlinear damping force component and the nonlinear elastic characteristic. The ordinary differential equation of the motion can be written as:

$$m\ddot{\eta}(t) + g(\eta(t), \dot{\eta}(t)) = F(t) \,, \tag{1}$$

where m is the mass, $F(t)$ is the external excitation signal with zero mean, and $\eta(t)$ is the displacement response of the system.
The equation of motion can be rewritten as:

$$\ddot{\eta}(t) + h(\eta(t), \dot{\eta}(t)) = f(t) \,. \tag{2}$$

The idea of linearization is replacing the equation by a linear system:

$$\ddot{\eta}(t) + \beta_{ech}\,\dot{\eta}(t) + \gamma_{ech}\eta(t) = f(t) \,. \tag{3}$$

It is necessary to minimize the expected value [1,2] of the difference between equations (2) and (3) in a least square sense. Now, the difference is the difference between the nonlinear stiffness and linear stiffness terms , which is

$$\varepsilon = h(\eta(t),\dot{\eta}(t)) - \beta_{ech}\,\dot{\eta}(t) - \gamma_{ech}\eta(t)\,. \tag{4}$$

The value of β_{ech} and γ_{ech} can be obtained by minimizing [3,4] the expectation of the square error

$$\frac{\partial}{\partial\,\beta_{ech}}E[\varepsilon^2]=0 \tag{5}$$

and

$$\frac{\partial}{\partial\,\gamma_{ech}}E[\varepsilon^2]=0\,. \tag{6}$$

Because

$$E\{\varepsilon^2\}=E\{h^2\}+\beta_{ech}{}^2E\{\dot{\eta}^2\}+\gamma_{ech}{}^2E\{\eta^2\}-2\beta_{ech}E\{\dot{\eta}\,h\}+2\beta_{ech}\gamma_{ech}E\{\dot{\eta}\,\eta\}-2\gamma_{ech}E\{\eta h\} \tag{7}$$

we obtain

$$E\{\dot{\eta}\,h\}-\beta_{ech}E\{\dot{\eta}^2\}-\gamma_{ech}E\{\eta\,\dot{\eta}\}=0\,, \tag{8}$$

$$E\{\eta h\}-\beta_{ech}E\{\eta\,\dot{\eta}\}-\gamma_{ech}E\{\eta^2\}=0\,. \tag{9}$$

The expression of β_{ech} can be obtained as:

$$\beta_{ech}=\frac{E\{\eta^2\}E\{\dot{\eta}\,h\}-E\{\eta\,\dot{\eta}\}E\{\eta h\}}{E\{\eta^2\}E\{\dot{\eta}^2\}-(E\{\eta\,\dot{\eta}\})^2}\,, \tag{10}$$

$$\gamma_{ech}=\frac{E\{\dot{\eta}^2\}E\{\eta h\}-E\{\eta\,\dot{\eta}\}E\{\dot{\eta}\,h\}}{E\{\eta^2\}E\{\dot{\eta}^2\}-(E\{\eta\,\dot{\eta}\})^2}\,. \tag{11}$$

Because

$$E\{\dot{\eta}\,\eta\}=0\,, \tag{12}$$

the linear equation for the random excitation is

$$\ddot{\eta}(t)+\frac{E\{\dot{\eta}\,h\}}{E\{\dot{\eta}^2\}}\dot{\eta}(t)+\frac{E\{\eta h\}}{E\{\eta^2\}}\eta(t)=f(t). \tag{13}$$

The displacement variance [5,6,7] of the system under Gaussian white noise excitation can be expressed as

$$\sigma_\eta^2=\frac{1}{m}\int_{-\infty}^{\infty}\frac{S_0}{\left(\frac{E\{\eta h\}}{E\{\eta^2\}}-\omega^2\right)^2+\omega^2\left(\frac{E\{\dot{\eta}\,h\}}{E\{\dot{\eta}^2\}}\right)^2}d\omega, \tag{14}$$

where the frequency response function [4,8,9] of the single degree of freedom system is

$$H(\omega)=\frac{1}{m\left(-\omega^2+i\omega\frac{E\{\dot{\eta}\,h\}}{E\{\dot{\eta}^2\}}+\frac{E\{\eta h\}}{E\{\eta^2\}}\right)}. \tag{15}$$

The power spectral density of the response [10,11,12] is

$$S_\eta(\omega)=\frac{S_F(\omega)}{m^2\left[\left(\frac{E\{\eta h\}}{E\{\eta^2\}}-\omega^2\right)^2+\omega^2\left(\frac{E\{\dot{\eta}\,h\}}{E\{\dot{\eta}^2\}}\right)^2\right]} \tag{16}$$

Taking into account that the integral type is

$$\int_{-\infty}^{\infty}\frac{\omega^2B_1{}^2+B_0{}^2}{\left(A_0-\omega^2A_2\right)^2+\omega^2A_1{}^2}d\omega=\pi\frac{\frac{B_0{}^2}{A_0}A_2+B_1{}^2}{A_1A_2}\,, \tag{17}$$

we obtains for the displacement variance [12,13]

$$\sigma^2_{\eta} = \pi S_0 \frac{E\{\eta^2\}E\{\dot{\eta}^2\}}{mE\{\eta h\}E\{\dot{\eta} h\}} . \tag{18}$$

We know

$$R_{\dot{\eta}}(\tau) = -\frac{d^2 R_\eta(\tau)}{d\tau^2} = \int_{-\infty}^{\infty} \omega^2 S_\eta(\omega) e^{i\omega\tau} d\omega . \tag{19}$$

We obtains

$$\begin{aligned} \sigma^2_{\dot{\eta}} &= E\{\dot{\eta}^2\} = R_{\dot{\eta}}(0) = \int_{-\infty}^{\infty} \omega^2 S_\eta(\omega) d\omega = \int_{-\infty}^{\infty} \omega^2 \left| H(\omega) \right|^2 S_F d\omega = \\ &= \int_{-\infty}^{\infty} \omega^2 \left| H(\omega) \right|^2 m S_0 d\omega = \frac{1}{m} \int_{-\infty}^{\infty} \omega^2 \left| H(\omega) \right|^2 S_0 d\omega. \end{aligned} \tag{20}$$

The velocity variance is:

$$\sigma^2_{\dot{\eta}} = \frac{1}{m} \int_{-\infty}^{\infty} \frac{\omega^2 S_0}{\left(\frac{E\{\eta h\}}{E\{\eta^2\}} - \omega^2 \right)^2 + \omega^2 \left(\frac{E\{\dot{\eta} h\}}{E\{\dot{\eta}^2\}} \right)^2} d\omega = \pi S_0 \frac{E\{\dot{\eta}^2\}}{mE\{\dot{\eta} h\}} . \tag{21}$$

2. EXAMPLE: THE RANDOM DUFFING OSCILLATOR

For convenience, the Duffing oscillator has been used to illustrate this procedure here. Consider the Duffing equation of motion:

$$\ddot{\eta}(t) + 2\xi pc[\dot{\eta}(t) + \varepsilon r_3 \dot{\eta}^3(t)] + p^2 \eta(t) + p^2 \alpha \eta^3(t) = f(t) , \tag{22}$$

with parameters $m=1kg,\ k=25\frac{N}{m},\ r_3=3\cdot 10^3 s^2/m^2,\ c=1,5\frac{N\cdot s}{m},\ \varepsilon=0,01$ and the spectral density for excitation $S_F = 1N^2 \cdot s$.

Obtain in this case for the displacement variance

$$\sigma^2_{\eta} = \frac{\pi S_0}{m} \frac{1}{2\xi p^3(1+3\alpha\sigma^2_\eta)\left[1+\varepsilon\left(\frac{63}{4} r_3 \sigma^2_{\eta}\right)\right]} . \tag{23}$$

The damping characteristic is

$$\beta_{ech} = 2\xi_e p_e = 2\xi p \left[1+\varepsilon\left(\frac{63}{4} r_3 \sigma^2_{\eta}\right)\right]. \tag{24}$$

The elastic characteristic is given by

$$\gamma_{ech} = p_e^2 = p^2(1+3\alpha\sigma^2_\eta) . \tag{25}$$

The linear equation for the random excitation is

$$\ddot{\eta}(t) + 2\xi p \left[1+\varepsilon\left(\frac{63}{4} r_3 \sigma^2_{\eta}\right)\right] \dot{\eta}(t) + p^2(1+3\alpha\sigma^2_\eta)\eta(t) = f(t) . \tag{26}$$

The frequency response function [11,13] of the single degree of freedom system is

$$\left| H(\omega) \right| = \frac{1}{m\sqrt{\left(p^2 + 3\alpha p^2 \sigma^2_\eta - \omega^2\right)^2 + 4\xi^2 p^2 \omega^2 \left[1+\varepsilon\left(\frac{63}{4} r_3 \sigma^2_{\eta}\right)\right]^2}} . \tag{27}$$

The displacement variance of the system under Gaussian white noise excitation can be expressed as

$$\sigma^2_{\eta} = \pi S_0 \frac{1}{2\xi p^3(1+3\alpha\sigma^2_{\eta})\left[1+\varepsilon\left(\frac{63}{4} r_3 \sigma^2_{\eta}\right)\right]} . \tag{28}$$

The resulting solution

$$\sigma^2_{\eta} = 0,02 m^2 . \tag{29}$$

The velocity variance is given by

$$\sigma^2_{\dot{\eta}} = p^2 \sigma^2_{\eta} (1+3\alpha\sigma^2_{\eta}) = 71\cdot 10^{-2} \frac{m^2}{s^2} . \tag{30}$$

The power spectral density of the response is

$$S_{\eta}(\omega) = \frac{S_0}{\left(35,4-\omega^2\right)^2 + 17,82\omega^2} . \tag{31}$$

In figure 1. the power spectral density of the response are given for various combinations of the parameter.

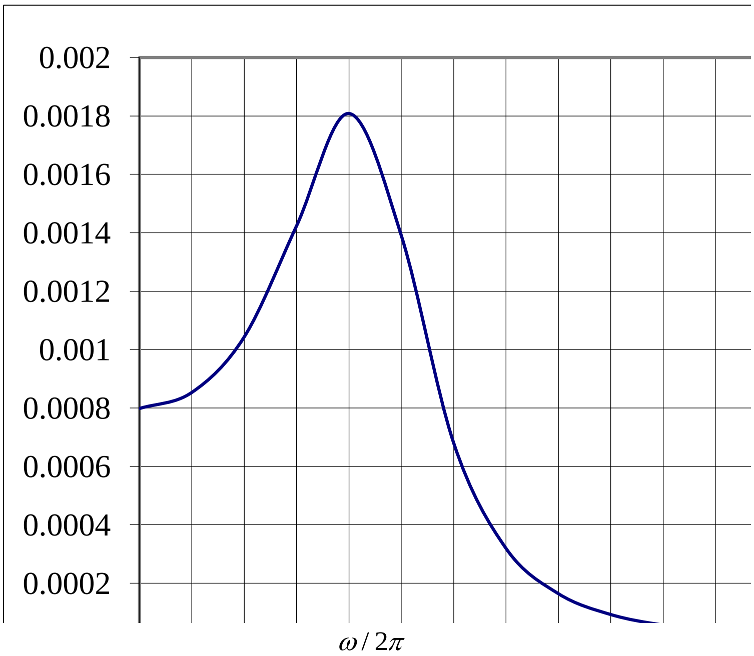

Fig. 1 The power of spectral density $S_{\eta}[m^2 \cdot s]$ for

$m = 1kg, k = 25\frac{N}{m}, c = 1,5\frac{Ns}{m}$.

3. CONCLUSION

Detailed numerical results are presented for of nonlinear oscillators under white noise excitation. Note that the maximum spectral power density values are obtained for velocity.0,8 rad/s. Increases are pronounced in the frequency 0,4…0,8 rad/s and then slowly declines occurring in frequency band 1,6…2,4 rad/s.

Efficient equivalent linear systems with random coefficients for approximating the power spectral density can be deduced. The resonant peak is described very satisfactorily by the approximate solution.

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