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Three points frictionless simultaneous collision of rigid solid

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Abstract: - *This paper deals with the simultaneous frictionless three-point collision of a rigid solid. The working hypotheses and the conditions under which the problem can be solved are presented. The impulses at the contact points, the velocity of the rigid body after the collision and its energy variation are determined. Some special cases are also discussed. The theory is illustrated based on a fully solved example. The paper concludes with conclusions and future directions for study.*

Keywords: *Collision, simultaneous, multi-point, coefficient of restitution*

1. INTRODUCTION

The actual study of the problem is presented in our previous paper [26], based on the references [1 – 25]. Some aspects must be remembered here [26]:

– simultaneous vanishing of the normal velocities in the contact points;

– there is no jamb phenomenon.

In the previous paper we have discussed some aspects concerning the simultaneous collisions. In this paper we consider two rigid bodies with constraints which simultaneously collide at three points.

2. TECHNICAL REQUIREMENTS

We will use the same notations as in [26].

In a similar way to the simultaneous collisions at two points of a rigid solid with bilateral contraints, one may write

$$\{\mathbf{v}^{(1)}\}-\{\mathbf{v}^{0(1)}\} = [\mathbf{Q}_1][\mathbf{M}_{1red}]^{-1}[\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_1]\{\mathbf{P}\}, \qquad (1)$$

for the first rigid solid, and

$$\{\mathbf{v}^{(2)}\} - \{\mathbf{v}^{0(2)}\} = [\mathbf{Q}_2][\mathbf{M}_{2red}]^{-1}[\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_2]\{\mathbf{P}\}, \qquad (2)$$

for the second one.

The previous expressions are multiplied at the left side by $[\mathbf{U}_1]^{\mathrm{T}}[\boldsymbol{\eta}]$ and $[\mathbf{U}_2]^{\mathrm{T}}[\boldsymbol{\eta}]$ obtaining

$$\{\mathbf{v}_n^{(1)}\} - \{\mathbf{v}_n^{0(1)}\} = [\mathbf{U}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_1][\mathbf{M}_{1red}]^{-1}[\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_1]\{\mathbf{P}\}, \qquad (3)$$

$$\{\mathbf{v}_n^{(2)}\} - \{\mathbf{v}_n^{0(2)}\} = [\mathbf{U}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_2][\mathbf{M}_{2red}]^{-1}[\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_2]\{\mathbf{P}\}. \qquad (4)$$

It results

$$[\mathbf{U}_i]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_i][\mathbf{M}_{ired}]^{-1}[\mathbf{Q}_i]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_i] = [\mathbf{G}_i], \quad i = 1, 2, \qquad (5)$$

where

$$\{\mathbf{v}_{12n}\} - \{\mathbf{v}_{12n}^{0}\} = -[[\mathbf{G}_1] + [\mathbf{G}_2]]\{\mathbf{P}\}. \qquad (6)$$

In addition, one has

$$\{\mathbf{v}_{12n}\} - \{\mathbf{v}_{12n}^{0}\} = -[[\mathbf{I}] + [\mathbf{K}]]\{\mathbf{v}_{12n}^{0}\}, \qquad (7)$$

with

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad (8)$$

and

$$[\mathbf{K}] = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}. \qquad (9)$$

We also get

$$\{\mathbf{P}\} = [[\mathbf{G}_1] + [\mathbf{G}_2]]^{-1}[[\mathbf{I}] + [\mathbf{K}]]\{\mathbf{v}_{12n}^{0}\}. \qquad (10)$$

where

$$[\mathbf{M}_{ired}] = [\mathbf{Q}_i]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_i][\mathbf{Q}_i], \quad i = 1, 2, \qquad (11)$$

$$\left\{\mathbf{v}_n^{0(i)}\right\} = \left[\mathbf{U}_i\right]^{\mathrm{T}}\left[\boldsymbol{\eta}\right]\left\{\mathbf{v}^{0(i)}\right\}, \ i = 1, 2, \qquad (12)$$

$$\left\{\mathbf{v}_{12n}^{0}\right\} = \left\{\mathbf{v}_n^{0(1)}\right\} - \left\{\mathbf{v}_n^{0(2)}\right\}. \qquad (13)$$

The velocities after the collision are

$$\left\{\mathbf{v}^{(1)}\right\} = \left\{\mathbf{v}^{0(1)}\right\} - \left[\mathbf{Q}_1\right]\left[\mathbf{M}_{1red}\right]^{-1}\left[\mathbf{Q}_1\right]^{\mathrm{T}}\left[\boldsymbol{\eta}\right]\left[\mathbf{U}_1\right]\left\{\mathbf{P}\right\} \qquad (14)$$

and

$$\left\{\mathbf{v}^{(2)}\right\} = \left\{\mathbf{v}^{0(2)}\right\} + \left[\mathbf{Q}_2\right]\left[\mathbf{M}_{2red}\right]^{-1}\left[\mathbf{Q}_2\right]^{\mathrm{T}}\left[\boldsymbol{\eta}\right]\left[\mathbf{U}_2\right]\left\{\mathbf{P}\right\}. \qquad (15)$$

The impulses of constraints are

$$\left\{\boldsymbol{\xi}_1\right\} = \left[\left[\mathbf{S}_1\right]^{\mathrm{T}}\left[\boldsymbol{\eta}\right]\left[\mathbf{M}_1\right]^{-1}\left[\mathbf{S}_1\right]\right]^{-1} \left[\mathbf{S}_1\right]^{\mathrm{T}}\left[\boldsymbol{\eta}\right]\left[\mathbf{M}_1\right]^{-1}\left[\mathbf{U}_1\right]\left\{\mathbf{P}\right\} \qquad (16)$$

and

$$\left\{\boldsymbol{\xi}_2\right\} = -\left[\left[\mathbf{S}_2\right]^{\mathrm{T}}\left[\boldsymbol{\eta}\right]\left[\mathbf{M}_2\right]^{-1}\left[\mathbf{S}_2\right]\right]^{-1}. \left[\mathbf{S}_2\right]^{\mathrm{T}}\left[\boldsymbol{\eta}\right]\left[\mathbf{M}_2\right]^{-1}\left[\mathbf{U}_2\right]\left\{\mathbf{P}\right\}. \qquad (17)$$

The matrices of the simple impulses are $\left[\mathbf{S}_i\right]\left\{\boldsymbol{\xi}_i\right\}$, with $i = 1, 2$.

Example 1. For the frames in Fig. 1, jointed at the points O_1 and O_2, have the centers of weight in C_1 and C_2 and collide at the points A_1, A_2 and A_3, one knows the dimension a, the masses m_1 and m_2, the inertial moments J_{x_1}, J_{y_1}, J_{z_1}, and J_{x_2}, J_{y_2}, J_{z_2},respecively, relative to each frame with respect to the central inertial systems, the coefficients of restitution k_1 at the point A_1. k_2 at the point A_2, and k_3 at the point A_3, and the initial distributions of velocities (the magnitudes of the angular veocities ω_{10} and ω_{20}).

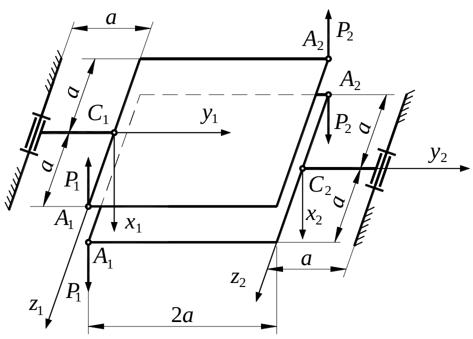

Figure 1. Example 1

One considers that the frames are identical: $m_1 = m_2$, $J_{x_1} = J_{x_2} = J_x$, $J_{y_1} = J_{y_2} = J_y$, $J_{z_1} = J_{z_2} = J_z$, $\omega_{10} = \omega_{20} = \omega_0$,

One asks for the impulses at the collision points A_1, A_2, and A_3, the constraint impulses at the points O_1 and O_2, and the distributions of velocities after the collision.

We get:

$$\mathbf{C_1O_1} \times \mathbf{i_1} = a\mathbf{k}\text{, } \mathbf{C_1O_1} \times \mathbf{j_1} = \mathbf{0}\text{, } \mathbf{C_1O_1} \times \mathbf{k} = -a\mathbf{i}\text{,}$$

$$[\mathbf{S}_1] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -a & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a & 0 & 0 & 0 & 0 \end{bmatrix}\text{,}$$

$$[\mathbf{Q}_1] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -a \\ 0 \\ 0 \end{bmatrix}\text{, } [\boldsymbol{\eta}][\mathbf{Q}_1] = \begin{bmatrix} -a \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\text{,}$$

$$[\mathbf{S}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_1] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}\text{, } [\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{S}_1] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}\text{,}$$

$$\mathbf{C_2O_2} \times \mathbf{i_2} = -a\mathbf{k}\text{, } \mathbf{C_2O_2} \times \mathbf{j_2} = \mathbf{0}\text{, } \mathbf{C_2O_2} \times \mathbf{k_2} = a\mathbf{i}\text{,}$$

$$[\mathbf{S}_2] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a & 0 & 0 & 0 & 0 \end{bmatrix}\text{,}$$

$$[\mathbf{Q}_2] = \begin{bmatrix} 0 \\ 0 \\ 1 \\ a \\ 0 \\ 0 \end{bmatrix}\text{, } [\boldsymbol{\eta}][\mathbf{Q}_2] = \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\text{,}$$

$$[\mathbf{S}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}\text{, } [\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{S}_2] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}\text{,}$$

$$\mathbf{C_1A_1} \times \mathbf{i_1} = a\mathbf{j}\text{, } \mathbf{C_1A_2} \times \mathbf{i_1} = a\mathbf{j} - 2a\mathbf{k}\text{, } \mathbf{C_1A_3} \times \mathbf{i_1} = -a\mathbf{j}\text{,}$$

$$\mathbf{C_2A_1} \times \mathbf{i_2} = a\mathbf{j} + 2a\mathbf{k}\text{, } \mathbf{C_2A_2} \times \mathbf{i_2} = -a\mathbf{j}\text{, } \mathbf{C_2A_3} \times \mathbf{i_2} = -a\mathbf{j} + 2a\mathbf{k}\text{,}$$

$$[\mathbf{U}_1]=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ a & a & -a\\ 0 & -2a & 0\end{bmatrix},\ [\mathbf{U}_2]=\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ a & -a & -a\\ 2a & 0 & 2a\end{bmatrix},$$

$$[\mathbf{M}_{1red}]=[\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_1][\mathbf{Q}_1]=m_1a^2+J_{z_1},\ [\mathbf{M}_{2red}]=[\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{M}_2][\mathbf{Q}_2]=m_2a^2+J_{z_2},$$

$$\{\mathbf{v}^{0(1)}\}=[0\quad 0\quad -\omega_{10}\quad a\omega_{10}\quad 0\quad 0]^{\mathrm{T}},\ \{\mathbf{v}^{0(2)}\}=[0\quad 0\quad -\omega_{20}\quad -a\omega_{20}\quad 0\quad 0]^{\mathrm{T}},$$

$$\{\mathbf{v}_{1n}^0\}=[\mathbf{U}_1]^{\mathrm{T}}[\boldsymbol{\eta}]\{\mathbf{v}^{0(1)}\}=\begin{bmatrix}a\omega_{10}\\ 3a\omega_{10}\\ a\omega_{10}\end{bmatrix},\ \{\mathbf{v}_{2n}^0\}=[\mathbf{U}_2]^{\mathrm{T}}[\boldsymbol{\eta}]\{\mathbf{v}^{0(2)}\}=\begin{bmatrix}-3a\omega_{20}\\ -a\omega_{20}\\ -3a\omega_{20}\end{bmatrix},$$

$$\{\mathbf{v}_{12n}^0\}=\{\mathbf{v}_{1n}^0\}-\{\mathbf{v}_{2n}^0\}=\begin{bmatrix}a\omega_{10}+3a\omega_{20}\\ 3a\omega_{10}+a\omega_{20}\\ a\omega_{10}+3a\omega_{20}\end{bmatrix},$$

$$[\mathbf{G}_1]=[\mathbf{U}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_1][\mathbf{M}_{1red}]^{-1}[\mathbf{Q}_1]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_1]$$
$$=\frac{a^2}{m_1a^2+J_{z_1}}\begin{bmatrix}1 & 3 & 1\\ 3 & 9 & 3\\ 1 & 3 & 1\end{bmatrix},$$

$$[\mathbf{G}_2]=[\mathbf{U}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{Q}_2][\mathbf{M}_{2red}]^{-1}[\mathbf{Q}_2]^{\mathrm{T}}[\boldsymbol{\eta}][\mathbf{U}_2]$$
$$=\frac{a^2}{m_2a^2+J_{z_2}}\begin{bmatrix}9 & 3 & 9\\ 3 & 1 & 3\\ 9 & 3 & 9\end{bmatrix}.$$

It results

$$[\mathbf{G}_1]+[\mathbf{G}_2]=\frac{a^2}{m_1a^2+J_{z_1}}\begin{bmatrix}1 & 3 & 1\\ 3 & 9 & 3\\ 1 & 3 & 1\end{bmatrix}$$
$$+\frac{a^2}{m_2a^2+J_{z_2}}\begin{bmatrix}9 & 3 & 9\\ 3 & 1 & 3\\ 9 & 3 & 9\end{bmatrix}$$

and

$$\det([G_1])=0,\ \det([G_2])=0,\ \det([G_1]+[G_2])=0,$$

that is, the matrix $[G_1]+[G_2]$ is not an invertible one and, consequently, the rest of the parameters can not be determined. In fact, the problem can not be solved if the three impulses are coplanar, conqurent or paralel (our case)

The problem can be solved if we change the direction of impulse P_3 as we present in Fig. 3.2.

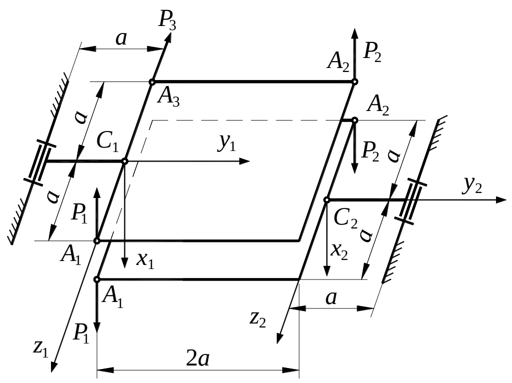

Figure 2. Example 2

3. CONCLUSIONS

In this paper we discuss the simultaneous multi-points collision of two rigid bodies with constraints. The collision takes place at three different points. The conditions for the solving of problem are also described and they took into account the positions of the three impulses. Based on it, the first example given in the paper can not be solved because it leads to a non-invertible matrix. The second example which avoids this inconvenient can be completely solved.

The method can be easily generalize to a simultaneous collision of two constrained rigid bodies at an arbitrary number of points.

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