SOLUTION ALGORITHMS FOR FINITE ELEMENT STRUCTURAL ANALYSIS EMPLOYING PRECONDITIONED CONJUGATE GRADIENT METHODS

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This paper summarizes the experience of the authors in use of conjugate gradient algorithms in finite element stuctural analysis. There are treated both linear static and dynamic problems. The solver for systems of linear equations corresponding to the static problem, and the extraction of eigenvalues and the corresponding eigenvectors in the case of vibration analysis of structures are based on conjugate gradient algorithms. These iterative methods were coupled with a preconditioning using an approximate Choleski decomposition. The computer program modules are using sparse storage technics for matrix operations, and they were incorporated in a PC version of the SAP V Liege finite element analysis program. The analysis of several structures by these methods shows an improvement of performances versus direct methods. The precision of numerical solving increase, while the computer time and storage requirements are diminished. These solvers are very advantageous especially in case of large structures.

1. INTRODUCTION

The solution algorithms presented in this paper are designed for the finite element analysis of elastic structures.

The static analysis of the structures leads to the solution of a set of linear simultaneous equations as a key part of the computation.

$$Au = b \ , \tag{1}$$

where the stiffness matrix A is a symmetric, positive definite, and sparse (nxn) matrix, u is the vector of nodal displacements, and b is the vector of nodal loadings.

In the analysis of structures only a few of smallest eigenvalues $\lambda = \omega^2$ of the problem

$$Au = \lambda Bu \ , \tag{2}$$

are to be calculated together with their eigenvectors u .The global mass matrix B is also symmetric and sparse.

2. THE PRECONDITIONED CONJUGATE GRADIENT METHOD FOR LINEAR EQUATIONS

The solution algorithm for the matrix equation (1) is ([2],[5],[6],[7],[8],[9] :

$$u^0 \text{ ; starting vector ; } r^0 = b - Au^0 \text{ ; } p^0 = v^0 = W^{-1} r^0 \text{ ;} \tag{3}$$

$$q^k = A\,p^k \text{ ; } a_k = (r^k, p^k)/(p^k, q^k) \text{ ; } u^{k+1} = u^k + a_k p^k \text{ ; } r^{k+1} = r^k - a_k q^k \text{ ;}$$

$$v^{k+1} = W^{-1} r^{k+1} \text{ ; } b_k = (r^{k+1}, v^{k+1})/(r^k, v^k) \text{ ; } p^{k+1} = v^{k+1} + b_k p^k \ .$$

Here W is a symmetric, positive definite preconditionning matrix.
The speed of convergence of this algorithm depends on the choice of W. The best W approximates A, the best is the behaviour of this algorithm, but the compu-

tational work and storage required for the inversion of W increases.

2.1. The incomplete Choleski decomposition

In this paper is treated a preconditionning method by use of the incomplete Choleski decomposition ([6],[7],[8]). In this case

$$W = \tilde{L}\,\tilde{L}^T \tag{4}$$

where $\tilde{L}$ is a lower triangular matrix with the same sparsity pattern as the lower triangular part of A. By incomlete decomposition on obtain an approximative representation for A :

$$A = \tilde{L}\,\tilde{L}^T + E\ , \tag{5}$$

where E is an (nxn) error matrix that contains nonzero elements only out of the sparsity pattern of A. The exact decomposition $A=LL^T$ into lower and upper triangular factors, L and L^T, respectively, produce fill-in within most of the band or skyline and is hence costly. Te incomplete decomposition of large sparse matrices is less expensive because this fill-in is neglected.

Reordering of nodes

The ordering of nodes can affect the efficiency of the preconditioning. A k-colour reordering of nodes [5] before the incomplete decomposition improves the approximation. In this case the nodes of the same colour are not coupled one with other. For example, in the case of a regular plane frame structure (rectangular mesh with beam elements) results a two-colour reordering of nodes obtained by separating the mesh points into two sets, simulating the black and white squares of a chessboard and then numbering all of the nodes in one set before the nodes in the other set. For complexe types of structures the required number of colours can be greater. This type of reordering reduces the propagation of the effects of neglecting generated coefficients (fill-in) in the process of decomposition.

3. THE PRECONDITIONED CONJUGATE GRADIENT METHOD FOR EIGENVALUE PROBLEMS

The solution algorithm for the matrix equation (2) for the fundamental mode of vibration is :

$$u^o \text{ ; starting vector ; } u^o = u^o/\sqrt{(u^o, Bu^o)}\ \text{ ; } \lambda_o = (u^o, Au^o)\ \text{ ; } \tag{6}$$

$$g^o = Au^o - \lambda_o Bu^o \text{ ; } p^o = -v^o = -W^{-1}g^o \text{ ;}$$

Computes a_k from the equation :

$$(u^k, Ap^k) - \lambda_k(u^k, Bp^k) + a_k((p^k, Ap^k) - \lambda_k(p^k, Bp^k)) +$$

$$+ a_k^2\ ((p^k, Ap^k)(u^k, Bp^k) - (u^k, Ap^k)(p^k, Bp^k)) = 0 \text{ ;}$$

$$u^{k+1} = u^k + a_k p^k \text{ ; } u^{k+1} = u^{k+1}/\sqrt{(u^{k+1}, Bu^{k+1})} \text{ ; } \lambda_{k+1} = (u^{k+1}, Au^{k+1}) \text{ ;}$$

$$g^{k+1} = Au^{k+1} - \lambda_{k+1}Bu^{k+1} \text{ ; } v^{k+1} = W^{-1}g^{k+1} \text{ ; } b_k = (g^{k+1}, v^{k+1})/(g^k, v^k) \text{ ; } p^{k+1} = -v^{k+1} + b_k p^k \text{ .}$$

For the computation of higher modes the same algorithm is used, but in each step, the current approximation of the eigenvector u^i is M-othogonalized with the lower eigenvectors u^j (j=1,...,i-1) computed before [4]. This algorithm can be used even to the computation of structures with solid body degrees of freedom. The zero eigenvalues corresponding to this d.o.f. are approximed by very small numbers.

In the case of the same structure the solving of a load case by the algorithm (3) requires nearly the same computational work, that the solution of an eigenvector by the algorithm (6) [3].

4. NUMERICAL RESULTS

The algorithms presented above were imcorporated in a PC version of the SAP V Liege finite element analysis program. The computer program modules are using in-core and peripheral sparse storage technics and can handle very large matrices. The numerical results were obtained with a 12 Mhz AT-286 computer with mathematical coprocessor (80287).

In the table 1. is shown the behaviour of the algorithm (3) in the case of a square plate on elastic Winkler space. The structure was modeled by a grid of 4x4 shell and corresponding boundary elements. The stiffness of soil is given by the k factor. In the case of small k factors the displacement of the structure is close to the movement of a rigid body and the (3) algorithm becomes slow. A four-colour reordering of nodes shows a very good improvement in this case.

Table 1. Elastic supported plate - the effect of the ordering of nodes

	Number of required iterations for EPS=0.000001					
Stiffness of soil (k)	1.0	0.7	0.5	0.4	0.3	0.1
Lexicographical ordering	21	25	94	267	-	-
Four-colour ordering	10	10	10	10	11	11

In the table 2. is given a comparison between the algorithm (3) and the Gauss elimination algorithm for two 3D frame structures solved for one load case.

Table 2. Comparison between solvers for linear equations

Solution method	PCG	Elim.	PCG	Elim.
Number of degrees	750	750	1296	1296
Half bandwidth	-	156	-	222
Number of blocks	-	17	-	40
Number of iterations	44	-	50	-
Assembling time (seconds)	14	116	26	315
Solving time (seconds)	166	616	217	3150
Total time (seconds)	180	732	243	3465
External storage (K bytes)	96	499	174	1288

The table 3. presents a comparison between the subspace iteration algorithm and (6) for solving the first five vibration modes in the case of a 3D frame structure with 750 d.o.f.
The effective computing time was 1033 seconds in the first case and 3374 seconds in the case of the subspace iterations.

Tabelul 8. Comparison between eigenproblem solvers

Period (seconds)	mode 1	mode 2	mode 3	mode 4	mode 5
GCP	0.4760	0.4760	0.4655	0.1556	0.1543
SI	0.4760	0.4760	0.4655	0.1557	0.1543

5. CONCLUSION

This paper presents preconditioned conjugate gradient algorithms for the finite element analysis of structures. There are treated the systems of linear equations corresponding to the static problem and the solving of eigenvalue problems for the case of vibration analysis of structures.
The preconditioning by incomplete Choleski decomposition is very efficient. There is studied a reordering of nodes, which gives a further improvement of the computational speed.

In the case of solving linear systems of equations the comparison between the PCG method and Gauss elimination shows a reduction up to 14 times of the required solving time and up to 7 times of the external storage. This comparison was made for the same precision but in some cases the PCG method gives higher precision that direct methods. The comparison shows growing advantages for larger structures.

These methods are very efficient and permit the analysis of large structures with computers of low performance.

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