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A DUAL VECTORS BASED FORMALISM FOR PARAMETRIZATION OF RIGID BODY DISPLACEMENT AND MOTION

Daniel Condurache[1], Adrian Burlacu[2]
[1] Gheorghe Asachi Technical University of Iasi, Iasi, ROMANIA, daniel.condurache @gmail.com
[2] Gheorghe Asachi Technical University of Iasi, Iasi, ROMANIA, aburlacu@ac.tuiasi.ro

Abstract: *This paper reveals a set of dual vectors based methods for rigid body displacement and motion parameterization. When parameterization methods are designed, a very important objective is to obtain is a reduced number of algebraic equations and fewer variables for a more compact notation. This feature is achieved by acknowledging that parameterization of the rigid body motion is a problem strongly connected with the definition and properties of proper orthogonal dual tensors. Tensor analysis expresses the invariance of the laws of physics with respect to the change of basis and change of frame operations. First, we propose a method for computing orthogonal dual tensors based on the dual vectors derived from the motion laws of both points and lines attached to the rigid body. The rigid body motion parameterization using dual vectors gives the possibility of constructing new computational methods for the screw axis (SA) and instantaneous screw axis (ISA) motion parameters. These methods are based on two results: the computation of SA is equivalent to the computation of the logarithm of an orthogonal dual tensor, the computation of ISA is equivalent with the computation of an algebraic entity entitled "the velocity dual tensor".*
Keywords: *dual vector, rigid body, motion.*

1. INTRODUCTION

A rigid body can be characterized through different types of features, among them being points and lines. Starting with classical manipulator robot kinematics and dynamics description and finishing with the new results obtained in robotics, machine vision, astrodynamics or neuroscience, the range of applications involving points or lines transformation is very large [1-3]. If points are considered then any coordinates transformation can be parametrized using homogeneous transformations. For line features, parametrization techniques were developed using the dual numbers theory [4-7]. The combination of dual numbers, dual vectors or dual matrices calculus with elements of screw theory generates different techniques for rigid body motion modeling [8-10]. Orthogonal dual tensors are a complete tool for computing rigid body displacement and motion parameters. A reduced number of algebraic equations and a more compact notation with fewer variables are two of the advantages of orthogonal dual tensor based parametrization methods. The first goal of our research is to give a more compact algebraic description regarding rigid body motion parametrization using tensors and to discuss the advantages over the methods involving dual matrices [8-14]. Our tensorial parametrization method is generated by the properties of the dyadic product between dual vectors. The second contribution represents a set of new computational methods for the screw axis (SA) and instantaneous screw axis (ISA) motion parameters.
The mathematical preliminaries and the notations are presented in section 2. Different algebraic sets were used to construct the parametrization methods proposed in this paper, their most important properties being detailed in the appendixes. The construction of the dual tensors module using the dyadic product between a basis of dual vectors and its reciprocal is discussed in section 3. Section 4 focuses on a new rigid body motion parameterization using bases of dual vectors. The construction of orthogonal dual tensors and screw and instantaneous screw parameters computation techniques are detailed. Also, a short and constructive proof of the famous *Mozzi - Chasles* theorem using dual tensors is presented. Section 5 contains the conclusions and future work.

2 MATHEMATICAL PRELIMINARES AND NOTATIONS

This section outlines briefly the notations used in the rest of the paper and the algebraic properties of dual numbers, dual vectors and dual tensors. Regarding notation, in order to avoid name clashes, the following are considered: $\underline{x}$ denotes a dual number, $\underline{\mathbf{x}}$ a dual vector, ε represents the imaginary entity which fulfills $\varepsilon^2 = 0$. Details over the dual numbers, dual vectors and dual tensors sets can be found in [7], [14], [15].

• ***Dual numbers***

Let the set of real dual numbers be denoted by:

$$\underline{\mathrm{R}} = \mathrm{R} + \varepsilon \mathrm{R} = \{\underline{a} = a + \varepsilon a_0 \mid a, a_0 \in \mathrm{R}, \varepsilon^2 = 0\}. \tag{1}$$

where $a = Re(\underline{a})$ is the real part of $\underline{a}$ and $a_0 = Du(\underline{a})$ the dual part.

Any differentiable function of a dual number variable $\underline{x} = x + \varepsilon x_0$ can be decomposed as:

$$f(\underline{x}) = f(x) + \varepsilon x_0 f'(x). \tag{2}$$

The inverse of $\underline{a} \in \underline{\mathrm{R}}$, denoted by $\underline{a}^{-1} \in \underline{\mathrm{R}}$, exists if and only if $Re(\underline{a}) \neq 0$ and is computed using $\underline{a}^{-1} = \frac{1}{\underline{a}} = \frac{1}{a} - \varepsilon \frac{a_0}{a^2}$. Also, $\underline{a} \in \underline{\mathrm{R}}$ is a zero divisor if and only if $Re(\underline{a}) = 0$.

• ***Dual vectors***

In the Euclidean space, the linear space of free vectors with dimension 3 will be denoted by V_3. The ensemble of dual vectors is defined as

$$\underline{V}_3 = V_3 + \varepsilon V_3 = \{\underline{\mathbf{a}} = \mathbf{a} + \varepsilon \mathbf{a}_0 ; \mathbf{a}, \mathbf{a}_0 \in V_3, \varepsilon^2 = 0\}, \tag{3}$$

where $\mathbf{a} = Re(\underline{\mathbf{a}})$ is the real part of $\underline{\mathbf{a}}$ and $\mathbf{a}_0 = Du(\underline{\mathbf{a}})$ the dual part. For dual vectors, three products will be considered: scalar product (denoted by $\underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$), cross product (denoted by $\underline{\mathbf{a}} \times \underline{\mathbf{b}}$) and triple scalar product (denoted by $<\underline{\mathbf{a}}, \underline{\mathbf{b}}, \underline{\mathbf{c}}>$).

The magnitude of $\underline{\mathbf{a}}$, denoted by $|\underline{\mathbf{a}}|$, is the dual number which fulfills $|\underline{\mathbf{a}}| \cdot |\underline{\mathbf{a}}| = \underline{\mathbf{a}} \cdot \underline{\mathbf{a}}$ and can be computed using

$$|\underline{\mathbf{a}}| = \begin{cases} \Vert \mathbf{a} \Vert + \varepsilon \dfrac{\mathbf{a}_0 \cdot \mathbf{a}}{\Vert \mathbf{a} \Vert}, & Re(\underline{\mathbf{a}}) \neq 0 \\ \varepsilon \Vert \mathbf{a}_0 \Vert, & Re(\underline{\mathbf{a}}) = 0 \end{cases}, \tag{4}$$

where $\Vert . \Vert$ is the Euclidean norm. If $|\underline{\mathbf{a}}| = 1$ then $\underline{\mathbf{a}}$ is called unit dual vector.

Thus, based on these properties results that $(\underline{\mathrm{R}}, +, \cdot)$ is a commutative and unitary ring and any element $\underline{a} \in \underline{\mathrm{R}}$ is either invertible or zero divisor, while $(\underline{V}_3, +, \cdot_{\underline{\mathrm{R}}})$ is a free $\underline{\mathrm{R}}$-module.

• ***Dual tensors***

An $\underline{\mathrm{R}}$-linear application of $\underline{V}_3$ into $\underline{V}_3$ is called an Euclidean dual tensor:

$$\begin{aligned} &T(\underline{\lambda}_1 \underline{\mathbf{v}}_1 + \underline{\lambda}_2 \underline{\mathbf{v}}_2) = \underline{\lambda}_1 T(\underline{\mathbf{v}}_1) + \underline{\lambda}_2 T(\underline{\mathbf{v}}_2), \\ &\quad \forall \underline{\lambda}_1, \underline{\lambda}_2 \in \underline{\mathrm{R}}, \forall \underline{\mathbf{v}}_1, \underline{\mathbf{v}}_2 \in \underline{\mathbf{V}}_3. \end{aligned} \tag{5}$$

From now on, the Euclidean dual tensor $\boldsymbol{T}$ will be shortly called dual tensor and $\mathbf{L}(\underline{V}_3, \underline{V}_3)$ will denote the free $\underline{\mathrm{R}}$-module of dual tensors. To the authors knowledge, the properties of $\mathbf{L}(\underline{V}_3, \underline{V}_3)$ can be found only in a few articles like.

3. DUAL TENSOR CONSTRUCTION USING DUAL VECTORS

The rigid body displacement and motion parameterization methods proposed in section 4 are based on the properties of dual tensors. Thus, the present section the design of the dual tensor set is discussed. The key of the chosen design is the combination between dual bases and the dyadic product of dual vectors [16, 17]. In order to set-up the base of the dual tensor construction technique, we first uncover some algebraic results for dual bases.

Theorem 1 *If* $\underline{\mathbf{a}} \in \underline{V}_3$ *then a dual number* $\underline{\lambda} \in \underline{\mathrm{R}}$ *and a unit dual vector* $\underline{\mathbf{u}} \in \underline{V}_3$ *exist in order to have* $\underline{\mathbf{a}} = \underline{\lambda}\, \underline{\mathbf{u}}$. *Also, if* $Re(\underline{\mathbf{a}}) \neq \mathbf{0}$ *then* $\underline{\lambda}$ *and* $\underline{\mathbf{u}}$ *are unique up to a sign change.*

Proof. If $\|.\|$ is the Euclidean norm then $\underline{\lambda} = \|\mathbf{a}\| + \varepsilon \frac{\mathbf{a}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|}$ and $\underline{\mathbf{u}} = \frac{\mathbf{a}}{\|\mathbf{a}\|} + \varepsilon \frac{\mathbf{a} \times (\mathbf{a}_0 \times \mathbf{a})}{\|\mathbf{a}\|^3}$ proves the theorem when $Re(\underline{\mathbf{a}}) \neq \mathbf{0}$. If $Re(\underline{\mathbf{a}}) = \mathbf{0}$ then $\underline{\lambda} = \varepsilon \|\mathbf{a}_0\|$ and $\underline{\mathbf{u}} = \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|} + \varepsilon \underline{\mathbf{v}} \times \frac{\mathbf{a}_0}{\|\mathbf{a}_0\|}$, $\forall \underline{\mathbf{v}} \in \underline{V}_3$.

The geometrical interpretation of Theorem 1 is that any dual vector $\underline{\mathbf{a}}$ from $\underline{V}_3$, with $Re(\underline{\mathbf{a}}) \neq \mathbf{0}$, can be associated with a *labeled* line in the Euclidean three dimensional space. The elements of the dual vector $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ give the direction of the line parametrized as Plucker coordinates [2,3], while the dual number $\underline{\lambda} = |\underline{\mathbf{a}}| = \|\mathbf{a}\| + \varepsilon \frac{\mathbf{a}_0 \cdot \mathbf{a}}{\|\mathbf{a}\|}$ represents the label. If $Re(\underline{\mathbf{a}}) = \mathbf{0}$ then the geometrical interpretation is a set of parallel lines described by $\mathbf{a}_0$ and labeled with $\underline{\lambda} = |\underline{\mathbf{a}}| = \varepsilon \|\mathbf{a}_0\|$. □

Definition 1 *A set of three dual vectors* $\mathbf{B} = \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3\}$ *will be called dual basis if the dual vectors are* $\underline{\mathbb{R}}$ *linear independent and also represent a span set for* $\underline{V}_3$.

Proposition 1 *If any three dual vectors* $\underline{\mathbf{e}}_k \in \underline{V}_3, k = \overline{1,3}$, *fulfill* $Re(<\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3>) \neq 0$ *then there are uniquely determined* $\{\underline{\mathbf{e}}^1, \underline{\mathbf{e}}^2, \underline{\mathbf{e}}^3\}$ *using the conditions* $\underline{\mathbf{e}}_i \cdot \underline{\mathbf{e}}^j = \delta_i^j, i, j = \overline{1,3}$, *where* δ_i^j *is the Kronecker symbol.*

Proof. Let $\{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3\}$ be a set constructed by the following rules:

$$\underline{\mathbf{e}}^1 = \frac{\underline{\mathbf{e}}_2 \times \underline{\mathbf{e}}_3}{<\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3>}, \; \underline{\mathbf{e}}^2 = \frac{\underline{\mathbf{e}}_3 \times \underline{\mathbf{e}}_1}{<\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3>}, \underline{\mathbf{e}}^3 = \frac{\underline{\mathbf{e}}_1 \times \underline{\mathbf{e}}_2}{<\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3>}. \quad (6)$$

Using (6) the conditions $\underline{\mathbf{e}}_i \cdot \underline{\mathbf{e}}^j = \delta_i^j, i, j = \overline{1,3}$ are fulfilled. □

Remark 1 *For a dual basis* $\mathbf{B} = \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3\}$, *the set* $\mathbf{B}^* = \{\underline{\mathbf{e}}^1, \underline{\mathbf{e}}^2, \underline{\mathbf{e}}^3\}$ *represents its reciprocal dual basis. The dual basis* $\mathbf{B}$ *coincides with* $\mathbf{B}^*$ *if and only if* $\mathbf{B}$ *is an orthonormal basis (aka* $\underline{\mathbf{e}}_i \cdot \underline{\mathbf{e}}_j = \delta_{ij}$*).*

Given two dual vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}} \in \underline{V}_3$, $\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}$ denotes a dual tensor called ***tensor (dyadic) product*** and is defined by:

$$\underline{\mathbf{a}} \otimes \underline{\mathbf{b}} : \underline{V}_3 \times \underline{V}_3 \to \underline{V}_3, \; (\underline{\mathbf{a}} \otimes \underline{\mathbf{b}})\underline{\mathbf{v}} = (\underline{\mathbf{v}} \cdot \underline{\mathbf{b}})\underline{\mathbf{a}}, \forall \underline{\mathbf{v}} \in \underline{V}_3. \quad (7)$$

An important property of (7) is that $(\underline{\mathbf{a}} \otimes \underline{\mathbf{b}})(\underline{\mathbf{c}} \otimes \underline{\mathbf{d}}) = (\underline{\mathbf{b}} \cdot \underline{\mathbf{c}})\underline{\mathbf{a}} \otimes \underline{\mathbf{d}}$. From this point on we uncover how the dyadic product can be used to construct a dual tensor.

Theorem 2 *The following statements are true:*

1. A dual tensor $T : \underline{V}_3 \to \underline{V}_3$ is uniquely determined by the values obtained after T is applied to the elements of the dual basis $\mathbf{B} = \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3\}$:

$$T = (T\underline{\mathbf{e}}_i) \otimes \underline{\mathbf{e}}^i. \quad (8)$$

2. The ensemble $\mathbf{L}(\underline{V}_3, \underline{V}_3)$ is a free $\underline{\mathbb{R}}$ - module of rank equal to 9.

Proof. Starting with (8) the Einstein's rule for mute indexes summation, when i varies from 1 to 3, will be used. Let $\underline{\mathbf{v}} \in \underline{V}_3$ be an arbitrary vector that has the following expression in the basis $\mathbf{B} = \{\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3\}$:

$$\underline{\mathbf{v}} = (\underline{\mathbf{v}} \cdot \underline{\mathbf{e}}^j)\underline{\mathbf{e}}_j. \quad (9)$$

Using (5) it results that

$$\begin{aligned} T\underline{\mathbf{v}} &= T[((\underline{\mathbf{v}} \cdot \underline{\mathbf{e}}^j)\underline{\mathbf{e}}_j)] = (\underline{\mathbf{v}} \cdot \underline{\mathbf{e}}^j)(T\underline{\mathbf{e}}_j) = \\ &= [(T\underline{\mathbf{e}}_j) \otimes \underline{\mathbf{e}}^j]\underline{\mathbf{v}} \end{aligned} \quad (10)$$

which proves the first part of the theorem □.

If T is a dual tensor then the dual vectors $T\underline{\mathbf{e}}_j, j = \overline{1,3}$ can be written as

$$T\underline{\mathbf{e}}_j = [\underline{\mathbf{e}}^i \cdot (T\underline{\mathbf{e}}_j)]\underline{\mathbf{e}}_i, i = \overline{1,3}. \quad (11)$$

Denoting with $T_j^i = \underline{\mathbf{e}}^i \cdot (T\underline{\mathbf{e}}_j)$, $T_j^i \in \underline{\mathbb{R}}$ and combining (14) with (11) generates

$$T = T_j^i \underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}^j, \quad (12)$$

which represents a linear combination of tensors $\left\{\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}^j\right\}_{i,j=\overline{1,3}}$ that is equivalent with a spanning set. The previous result, together with the remark (which can be easily proven) that $\left\{\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}^j\right\}_{i,j=\overline{1,3}}$ are $\underline{\mathrm{R}}$ linearly independent in $\mathbf{L}$, imply that $\left\{\underline{\mathbf{e}}_i \otimes \underline{\mathbf{e}}^j\right\}_{i,j=\overline{1,3}}$ is a basis in $\mathbf{L}(\underline{V}_3,\underline{V}_3)$ and $rank_{\underline{\mathrm{R}}}\mathbf{L}(\underline{V}_3,\underline{V}_3) = 9$.

For any dual vector $\underline{\mathbf{a}} \in \underline{V}_3$ the associated skew-symmetric dual tensor will be denoted by $\tilde{\underline{\mathbf{a}}}$ and defined by:

$$\tilde{\underline{\mathbf{a}}}\underline{\mathbf{b}} = \underline{\mathbf{a}} \times \underline{\mathbf{b}}, \forall \underline{\mathbf{b}} \in \underline{V}_3. \tag{13}$$

The set of skew-symmetric dual tensors is structured as a free $\underline{\mathrm{R}}$-module of dimension 3, module which is isomorph with $\underline{V}_3$. The following notation are considered $\underline{\mathbf{a}} = \mathrm{vect}\,\tilde{\underline{\mathbf{a}}}$, $\tilde{\underline{\mathbf{a}}} = \mathrm{spin}\,\underline{\mathbf{a}}$ [2].

For an arbitrary dual tensor T the following entities can be computed

$$\mathrm{sym}T = \frac{1}{2}[T + T^{\mathrm{T}}],\ \mathrm{skew}T = \frac{1}{2}[T - T^{\mathrm{T}}], \tag{14}$$

where "sym" is the symmetric part of the dual tensor and "skew" is its skew-symmetric part. Also, the axial dual vector and the trace of tensor $\boldsymbol{T}$ are given by:

$$\mathrm{vect}\ T = \mathrm{vect}\frac{1}{2}[T - T^{\mathrm{T}}],\quad \mathrm{trace}\ T = \frac{< T\underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3 > + < \underline{\mathbf{e}}_1, T\underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3 > + < \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, T\underline{\mathbf{e}}_3 >}{< \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \underline{\mathbf{e}}_3 >}. \tag{15}$$

Both vect$T$ and trace$T$ have the $\underline{\mathrm{R}}$ -linearity property: $\forall \underline{\lambda}_1, \underline{\lambda}_2 \in \underline{\mathrm{R}}, \forall T_1, T_2 \in \mathbf{L}(\underline{V}_3,\underline{V}_3)$

$$\begin{aligned} &\mathrm{vect}(\underline{\lambda}_1 T_1 + \underline{\lambda}_2 T_2) = \underline{\lambda}_1 \mathrm{vect}T_1 + \underline{\lambda}_2 \mathrm{vect}T_2 \\ &\mathrm{trace}(\underline{\lambda}_1 T_1 + \underline{\lambda}_2 T_2) = \underline{\lambda}_1 \mathrm{trace}T_1 + \underline{\lambda}_2 \mathrm{trace}T_2 \end{aligned}. \tag{16}$$

If the dual tensor defined by (7) is analyzed, the following results emerge: $(\underline{\mathbf{a}} \otimes \underline{\mathbf{b}})^{\mathrm{T}} = \underline{\mathbf{b}} \otimes \underline{\mathbf{a}}$, $\mathrm{vect}(\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}) = \frac{1}{2}(\underline{\mathbf{b}} \times \underline{\mathbf{a}})$ and $\mathrm{trace}(\underline{\mathbf{a}} \otimes \underline{\mathbf{b}}) = \underline{\mathbf{a}} \cdot \underline{\mathbf{b}}$. These results combined with (16), when T is given by (11), lead to:

$$T^{\mathrm{T}} = \underline{\mathbf{e}}^i \otimes (T\underline{\mathbf{e}}_i), \tag{17}$$

$$\mathrm{vect}T = \frac{1}{2}(T\underline{\mathbf{e}}_i) \times \underline{\mathbf{e}}^i, \tag{18}$$

$$\mathrm{trace}T = (T\underline{\mathbf{e}}_i) \cdot \underline{\mathbf{e}}^i. \tag{19}$$

In (31), (32), (33) the Einstein's rule for mute indexes summation has been used, where i varies from 1 to 3.

4. RIGID BODY MOTION AND DISPLACEMENT PARAMETERIZATION

In order to have a more intuitive view of the equations that will be used in this subsection, the following notations must be considered:

$$\{f : \mathrm{R} \to V_3\} = V_3^{\mathrm{R}}, \{f : \mathrm{R} \to SO_3\} = SO_3^{\mathrm{R}}. \tag{20}$$

In (20), SO_3 denotes the special orthogonal group of tensors [2]. The method proposed by the authors emerges from the remark that any rigid body motion can be modeled using elements from the set of proper orthogonal dual tensors denoted by

$$\underline{SO}_3 = \{R \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) \mid RR^T = I, \det R = 1\} \tag{21}$$

and time depending functions, which can be grouped in a set denoted by $\underline{SO}_3^{\mathrm{R}}$:

$$\{f : \mathrm{R} \to \underline{SO}_3\} = \underline{SO}_3^{\mathrm{R}}. \tag{22}$$

The internal structure of any dual tensor $R \in \underline{SO}_3$ is illustrated by the following result:

Theorem 4 *For any $R \in \underline{SO}_3^{\mathrm{R}}$, an unique decomposition is viable*

$$R = Q + \varepsilon \tilde{\rho} Q, \tag{23}$$

where $Q = Q(t) \in SO_3^{\mathrm{R}}$ and $\rho = \rho(t) \in V_3^{\mathrm{R}}$.

Based on Theorem 4, a representation of any dual tensor from $\underline{SO}_3$ can be given:

Theorem 5 *For any orthogonal dual tensor R defined as in (37), a dual number $\underline{\alpha} = \alpha + \varepsilon d$ and a dual unit* *vector* $\underline{\mathbf{u}} = \mathbf{u} + \varepsilon \mathbf{u}_0$ *exists in order to have the following expression*

$$R = I + \sin\underline{\alpha}\tilde{\underline{\mathbf{u}}} + (1 - \cos\underline{\alpha})\tilde{\underline{\mathbf{u}}}^2, \tag{24}$$

where $\mathbf{u}$ and α are recovered from the linear invariants of Q, while $d = \rho \cdot \mathbf{u}$ and $\mathbf{u}_0 = \frac{1}{2}\rho \times \mathbf{u} + \frac{1}{2}\cot\frac{\alpha}{2}\mathbf{u} \times (\rho \times \mathbf{u})$.

Remark 3 *For every choice of an orthogonal dual tensor* $R \in \underline{SO}_3$ *a unit dual vector* $\underline{\mathbf{u}} \in \underline{V}_3$ *exists so that* $R\underline{\mathbf{u}} = \underline{\mathbf{u}}$.

The fundamental Mozzi-Chasles [2] theorem states that: "any rigid displacement may be represented by a planar rotation about a suitable axis passing through that point, followed by a translation along that axis". Theorem 5 and Remark 3 are in fact the steps of a very short, elegant and constructive proof of this famous theorem. A screw axis is characterized by an unitary dual vector $\underline{\mathbf{u}}$ and the screw parameters (angle of rotation about the screw and the translation along the screw axis) structured an a dual angle $\underline{\alpha}$. For the following results, lets recall that the $\underline{SO}_3$ is a Lie group and its Lie algebra can be identified by the skew-symmetric dual tensors set $\underline{so}_3$.

Theorem 6 *If the skew-symmetric dual tensors set is denoted* $\underline{so}_3 = \{\underline{\alpha} \in \mathbf{L}(\underline{\mathbf{V}}_3, \underline{\mathbf{V}}_3) \,|\, \underline{\alpha} = -\underline{\alpha}^T\}$ *then the mapping*

$$\exp : \underline{so}_3 \to \underline{SO}_3, \exp(\underline{\alpha}) = e^{\underline{\alpha}} = \sum_{k=0}^{\infty} \frac{\underline{\alpha}^k}{k!} \tag{25}$$

is well defined and surjective.

The screw parameters computation are linked with the problem of finding the logarithm of an orthogonal dual tensor R, which is defined by

$$\log : \underline{SO}_3 \to \underline{so}_3, logR = \left\{\psi \in \underline{so}_3 \,|\, exp(\psi) = R\right\} \tag{26}$$

and is the inverse of (25). If the dual vector $\underline{\boldsymbol{\psi}} = \boldsymbol{\psi} + \varepsilon\boldsymbol{\psi}_0$ is computed as $\underline{\boldsymbol{\psi}} = \text{vect}(\psi)$ then using Theorem 1 results that $\underline{\boldsymbol{\psi}} = \underline{\alpha} \cdot \underline{\mathbf{u}}$, where $\underline{\alpha} = \| \quad + \varepsilon \frac{\boldsymbol{\psi} \quad _0}{\|}$ and $\underline{\mathbf{u}} = \frac{\boldsymbol{\psi}}{\|} + \varepsilon \frac{\boldsymbol{\psi} \quad _0}{\| \quad ^3}$. This result implies that $\underline{\boldsymbol{\psi}}$ can be used to parametrize any type of rigid motion. Before computing the logarithm of an orthogonal dual tensor, we need to analyze the behavior and influence of the tensor's natural invariants.

Remark 4 *A direct result of Theorems 5 and 6 is that the logarithm of a dual tensor is the product* $\underline{\alpha}\tilde{\underline{\mathbf{u}}}$. *The parameters* $\underline{\alpha}$ *and* $\underline{\mathbf{u}}$ *are called the natural invariants of* ***R*** *and can be recovered from the linear invariants [2] using (24):*

$$\underline{\mathbf{u}}\sin\underline{\alpha} = \text{vect}R, \tag{27}$$

$$1 + 2\cos\underline{\alpha} = \text{trace}R. \tag{28}$$

The above equations can be transformed into

$$\begin{aligned} &\underline{\mathbf{u}}\sin\underline{\alpha} = \text{vect}R \\ &\cos\underline{\alpha} = \frac{1}{2}[\text{trace}R - 1] \end{aligned} \tag{29}$$

and the parameters $\underline{\alpha}, \underline{\mathbf{u}}$. The unit dual vector $\underline{\mathbf{u}}$ gives the Plucker representation of the Mozzi-Chalses axis, while the dual angle $\underline{\alpha} = \alpha + \varepsilon d$ contains the rotation angle α and the translation distance d. The computational formulas for $\underline{\alpha}, \underline{\mathbf{u}}$ are extracted from (29):

$$\underline{\mathbf{u}} = \begin{cases} \pm\dfrac{\text{vect}R}{|\,\text{vect}R\,|}, & Re(\text{vectR}) \neq 0; \\[2ex] \dfrac{Q\mathbf{v}+\mathbf{v}}{\|\,Q\mathbf{v}+\mathbf{v}\,\|} + \varepsilon\dfrac{1}{2}\rho\times\dfrac{Q\mathbf{v}+\mathbf{v}}{\|\,Q\mathbf{v}+\mathbf{v}\,\|}, \forall\mathbf{v}\in V_3, & Re(\text{vectR}) = 0 \textit{ and } \text{trace}Q = -1; \\[2ex] \dfrac{\rho}{\|\,\rho\,\|}, & Re(\text{vectR}) = 0 \textit{ and } \text{trace}Q = 3; \end{cases} \tag{30}$$

$$\underline{\alpha} = atan2(\pm|\,\text{vect}R\,|, \frac{1}{2}[\text{trace}R - 1]). \tag{31}$$

The line containing the points of a rigid body undergoing minimum-magnitude velocities is called the instant screw axis (ISA) of the body under a given motion. The instantaneous motion of the body is equivalent to that of the bolt of a screw of ISA and is called instantaneous screw. An instantaneous screw axis, which will be defined as a dual vector denoted by $\underline{\boldsymbol{\omega}}$, is characterized by an unit dual vector $\underline{\mathbf{u}}$, a dual number $|\underline{\boldsymbol{\omega}}|$ called magnitude and a number p called the pitch.

Let $\underline{\mathbf{h}}_0$ embed the Plucker coordinates of a line at $t = t_0$ then:

$$\underline{\mathbf{h}}(t) = R(t)\underline{\mathbf{h}}_0. \tag{32}$$

Theorem 7 *In a general rigid motion, described by an orthogonal dual tensor, the* ***velocity dual tensor*** Φ *defined as*

$$\dot{\underline{\mathbf{h}}} = \Phi\underline{\mathbf{h}} \tag{33}$$

is expressed by:

$$\Phi = \dot{R}R^T. \tag{34}$$

The form of the velocity dual tensor described by (34) can be taken a step further if R is decomposed as in (23). This implies

$$\dot{R} = \dot{Q} + \varepsilon(\dot{\tilde{\rho}}Q + \tilde{\rho}\dot{Q}). \tag{35}$$

Because $\Phi \in \underline{so}_3^{\mathrm{R}}$ results that we can consider $\Phi = \tilde{\underline{\boldsymbol{\omega}}}$, which gives:

$$\tilde{\underline{\boldsymbol{\omega}}} = \dot{Q}Q^{\mathrm{T}} + \varepsilon(\dot{\tilde{\rho}} - \dot{Q}Q^{\mathrm{T}}\tilde{\rho}). \tag{36}$$

Let $\tilde{\boldsymbol{\omega}} = \dot{Q}Q^{\mathrm{T}}$ and $\tilde{\mathbf{v}} = \dot{\tilde{\rho}} - \dot{Q}Q^{\mathrm{T}}\tilde{\rho}$, then

$$\tilde{\underline{\boldsymbol{\omega}}} = \tilde{\boldsymbol{\omega}} + \varepsilon\tilde{\mathbf{v}}. \tag{37}$$

This equation leads to the internal structure of the dual vector $\underline{\boldsymbol{\omega}}$:

$$\underline{\boldsymbol{\omega}} = \boldsymbol{\omega} + \varepsilon\mathbf{v} \tag{38}$$

where $\boldsymbol{\omega}$ is the angular velocity and $\mathbf{v}$ represents the linear velocity of the point of the body that coincides instantaneously with the origin.

The dual vector $\underline{\omega}$ completely characterize, at a certain time, the velocity field of an rigid body in motion. Based on Theorem 1, for $\|\,\omega\,\| \neq 0$ the instantaneous screw axis unit dual vector is $\underline{\mathbf{u}} = \frac{\boldsymbol{\omega}}{\|} + \varepsilon\frac{\boldsymbol{\omega}}{\|\;^{3}}$ and $|\underline{\boldsymbol{\omega}}| = \|\,\boldsymbol{\omega}\,\| + \varepsilon\frac{\mathbf{v}\times}{\|}$. If $p = \frac{\mathbf{v}\times}{\|\;^{2}}$ denotes the pitch of the screw axis [9] then $|\underline{\boldsymbol{\omega}}| = \|\,\boldsymbol{\omega}\,\|(1+\varepsilon p)$. For $\|\,\boldsymbol{\omega}\,\| = 0$ we have an intantaneous pure translation.

5. CONCLUSIONS

The research presented in this paper is focused on developing a new rigid body motion parametrization method using dual vectors. Our studies showed that, in the dual tensors free module, the dual bases of dual vectors can completely characterize the rigid body motion from the Euclidean three-dimensional space. The proposed parametrization method was used to generate the orthogonal dual tensor that can model the motion of a rigid body. Screw and instantaneous screw parameters computational algorithms were also developed using dual bases of dual vectors. For the fundamental Mozzi-Chasles theorem a very short, elegant and constructive proof was provided. As future research goals the authors will analyze higher order kinematic properties using the free module of dual tensors. From the applicative point of view, the dual tensors algebra can be a solution for direct and inverse kinematics problems, multi-body problems, dual Euler-Rodrigues parameters and dual quaternions computation from direct measurements

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