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IN-PLANE VIBRATIONS OF PINNED-FIXED HETEROGENEOUS CURVED BEAMS UNDER A CONCENTRATED FORCE

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Abstract: *The paper deals with the vibrations of loaded heterogeneous curved beams when a centra load (a constant force) is exerted at the crown point of the beam. The effect of the loading is accounted by the strain it causes. It is assumed that the radius of curvature is constant and the Young's modulus and the Poisson's number are functions of the cross-sectional coordinates. The paper presents the determination of the Green function matrices for loads directed upwards and downwards. An appropriate numerical model is also provided, which makes possible to determine how the natural frequencies are related to the load. It is also shown that when the strain is zero, the corresponding formulae yield results valid for the free vibrations of curved beams.*

Keywords: *curved beams, heterogeneous material, natural frequency as a function of the load, Green function matrices*

1. INTRODUCTION

Curved beams are widely used in numerous engineering applications – let us consider, for instance, arch bridges, roof structures, or stiffeners in aerospace applications. Research into the mechanical behavior of such structural elements started in the 19^{th} century – see book [1] by Love for further details. The free vibrations of curved beams have been under extensive investigation: survey papers were published by Markus and Nanasi [2], Laura and Maurizi [3] as well as Chidampram and Lessia [4]. The PhD thesis by Szeidl [5] clarifies how the extensibility of the centerline affects the free vibrations and stability of circular beams subjected to a constant radial load (dead load) within the frames of the linear theory. The natural frequencies were computed by utilizing different numerical models. One of these relies on the Green function matrix of the corresponding boundary value problem. Unfortunately, the results of this work have not been published in English language. Paper [6] by Huang et al. takes shear deformations into account provided that the beam vibrates under the action of a constant vertical distributed load.

Lawther [7] investigates how a pre-stressed state of a body affects the natural frequencies. He studies finite dimensional multiparameter eigenvalue problems and finds that for multiparameter problems, the eigenvalue part of the solution is described by interaction curves in an eigenvalue space, and every such eigenvalue solution has a corresponding eigenvector. If all points on a curve have the same eigenvector, then the curve is necessarily a straight line, but the converse problem is far more complex. In the light of Lawther's results, there arises the question: how the frequencies change when a curved beam is subjected to a vertical force at the crown point. We assume that the curved beam is made of heterogeneous, isotropic and linearly elastic material. The cross-section is uniform in terms of both the geometry and the material composition. The material parameters are functions of the cross-sectional coordinates. Our main objectives: (1) derivation of those boundary value problems, which make possible to clarify how the load affects the natural frequencies; (2) determination of the Green function matrices, which can be used to reduce the eigenvalue problems set up for the natural frequencies to eigenvalue problems governed by systems of Fredholm integral equations; (3) reduction of the eigenvalue problems to algebraic ones and (4) the numerical solution of these.

2. THE PROBLEM FORMULATION

Figure 1 (a) shows a portion of the beam with the applied curvilinear coordinate system ($\xi = s, \eta, \zeta$) and (b) presents a pinned-fixed beam subjected to a load, directed downwards. By assumption, the uniform cross-section is symmetric with respect to the axis ζ. The Young's modulus and the Poisson number depend on the cross sectional coordinates in such a way that $E(\eta, \zeta) = E(-\eta, \zeta)$ and $\nu(\eta, \zeta) = \nu(-\eta, \zeta)$. Observe that the coordinate line $\xi = s$ coincides with the so-called

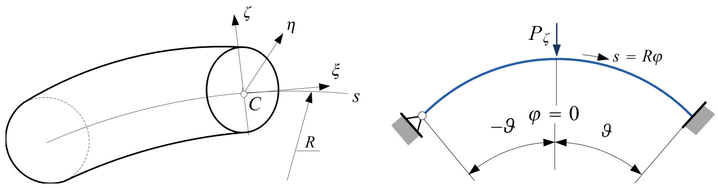

Figure 1. (a) Coordinate system, (b) Pinned-fixed beam

(E-weighted) centerline. This centerline intersects the cross-section at the point C. The location of the latter is obtained from the condition that the E-weighted first moment of the cross section with respect to the axis η is zero there:

$$Q_{e\eta} = \int_A E(\eta, \zeta)\, \zeta \,\mathrm{d}A = 0\,. \tag{1}$$

Let us now separate the load-induced, and otherwise time-independent mechanical quantities from those, which belong to the vibrations of the loaded beam. The latter ones are the time-dependent increments and are uniformly denoted by a subscript $_b$. Let u_o, w_o and R be the tangential and radial displacements and the radius of the centerline, respectively. Since this radius is constant, the coordinate line s and the angle coordinate φ are related to each other by $s = R\varphi$. The axial strain $\varepsilon_{o\xi}$ and the rigid body rotation $\psi_{o\eta}$ on the centerline can be expressed in terms of the displacements as

$$\varepsilon_{o\xi} = \frac{\mathrm{d}u_o}{\mathrm{d}s} + \frac{w_o}{R}, \qquad \psi_{o\eta} = \frac{u_o}{R} - \frac{\mathrm{d}w_o}{\mathrm{d}s}\,. \tag{2}$$

The principle of virtual work for a beam under distributed loading yields that the equilibrium equations

$$\frac{\mathrm{d}N}{\mathrm{d}s} + \frac{1}{R}\left[\frac{\mathrm{d}M}{\mathrm{d}s} - \left(N + \frac{M}{R}\right)\psi_{o\eta}\right] + f_t = 0\,, \quad \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{\mathrm{d}M}{\mathrm{d}s} - \left(N + \frac{M}{R}\right)\psi_{o\eta}\right] - \frac{N}{R} + f_n = 0 \tag{3a}$$

should be fulfilled by the axial force N and the bending moment M. Here f_t and f_n denote the intensity of the distributed loads on the centerline in the tangential and normal directions.

Hooke's law expresses the relation between the inner forces and the deformations [8] in such a way that

$$N = \frac{I_{e\eta}}{R^2}\varepsilon_{o\xi} - \frac{M}{R}\,, \qquad M = -I_{e\eta}\left(\frac{\mathrm{d}^2 w_o}{\mathrm{d}s^2} + \frac{w_o}{R^2}\right)\,, \quad N + \frac{M}{R} = \frac{I_{e\eta}}{R^2}\varepsilon_{o\xi}\,, \quad \text{where} \tag{4}$$

$$A_e = \int_A E(\eta,\zeta)\mathrm{d}A\,, \qquad I_{e\eta} = \int_A E(\eta,\zeta)\zeta^2\mathrm{d}A\,, \qquad m = \frac{A_e R^2}{I_{e\eta}} - 1 \tag{5}$$

A_e is referred to as the E-weighted area, $I_{e\eta}$ is the E-weighted moment of inertia with respect to the bending axis and m is a parameter. For the sake of brevity, we introduce dimensionless displacements and a notation for the derivatives:

$$U_o = \frac{u_o}{R}\,, \qquad W_o = \frac{w_o}{R}\,; \qquad (\ldots)^{(n)} = \frac{\mathrm{d}^n(\ldots)}{\mathrm{d}\varphi^n}\,, \quad n \in \mathbb{Z}\,. \tag{6}$$

Upon substitution of (4) and (2) into equilibrium equations (3), we obtain the following system of differential equations (DEs):

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 - m\varepsilon_{o\xi} \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(2)} + \\ + \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 + m(1 - \varepsilon_{o\xi}) \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix} = \frac{R^3}{I_{e\eta}}\begin{bmatrix} f_t \\ f_n \end{bmatrix}\,. \tag{7}$$

If we neglect the effects of the deformations to the equilibrium (so we set $\varepsilon_{o\xi}$ to zero), then we have

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(2)} + \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 + m \end{bmatrix}\begin{bmatrix} U_o \\ W_o \end{bmatrix} = \frac{R^3}{I_{e\eta}}\begin{bmatrix} f_t \\ f_n \end{bmatrix}\,. \tag{8}$$

With our notational conventions, quantities like the total tangential displacement is equal to the sum $u_o + u_{ob}$. It turns out that the increments in the axial strain and in the rotation have a very similar structure to equations (2):

$$\varepsilon_{mb} = \varepsilon_{o\xi\, b} + \psi_{o\eta}\psi_{o\eta\, b}\,, \qquad \psi_{o\eta\, b} = \frac{u_{ob}}{R} - \frac{\mathrm{d}w_{ob}}{\mathrm{d}s}\,, \qquad \varepsilon_{o\xi\, b} = \frac{\mathrm{d}u_{ob}}{\mathrm{d}s} + \frac{w_{ob}}{R}\,. \tag{9}$$

Based on the principle of virtual work, it can be shown that the equations of equilibrium with the increments are

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(N_b + \frac{M_b}{R}\right) - \frac{1}{R}\left(N + \frac{M}{R}\right)\psi_{o\eta\, b} + f_{tb} = 0\,, \tag{10a}$$

$$\frac{\mathrm{d}^2 M_b}{\mathrm{d}s^2} - \frac{N_b}{R} - \frac{\mathrm{d}}{\mathrm{d}s}\left[\left(N + \frac{M}{R}\right)\psi_{o\eta\,b} + \left(N_b + \frac{M_b}{R}\right)\psi_{o\eta}\right] + f_{nb} = 0\,. \tag{10b}$$

Due to the dynamical nature of the problem, the increments f_{tb} and f_{nb} are forces of inertia:

$$f_{tb} = -\rho_a A \frac{\partial^2 u_{ob}}{\partial t^2}, \qquad f_{nb} = -\rho_a A \frac{\partial^2 w_{ob}}{\partial t^2}\,. \tag{11}$$

Here A is the area of the cross-section, and ρ_a is the averaged density of the cross-section. Application of Hooke's law for the increments in the inner forces yields

$$N_b = \frac{I_{e\eta}}{R^2} m\varepsilon_{o\xi\,b} - \frac{M_b}{R}\,, \qquad M_b = -I_{e\eta}\left(\frac{\mathrm{d}^2 w_{ob}}{\mathrm{d}s^2} + \frac{w_{ob}}{R^2}\right), \qquad N_b + \frac{M_b}{R} = \frac{I_{e\eta}}{R^2} m\varepsilon_{o\xi\,b}\,. \tag{12a}$$

A comparison of equations (9), (10) and (12) result in the equations of motion:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 - m\varepsilon_{o\xi} \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix}^{(2)} +$$
$$+ \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 + m(1 - \varepsilon_{o\xi}) \end{bmatrix} \begin{bmatrix} U_{ob} \\ W_{ob} \end{bmatrix} = \frac{R^3}{I_{e\eta}} \begin{bmatrix} f_{tb} \\ f_{nb} \end{bmatrix}. \tag{13}$$

We remark that during the formal derivations we have neglected the quadratic term $\varepsilon_{o\xi}\varepsilon_{o\xi\,b}$ in (10a) and we have used the inequalities $\varepsilon_{o\xi\,b} \gg (\varepsilon_{o\xi\,b}\psi_{o\eta})^{(1)}$ and $1 \gg \varepsilon_{o\xi}$ in (10b), when utilizing Hooke's law. If the vibrations are assumed to be harmonic with the dimensionless displacement amplitudes $\hat{U}_{ob}$ and $\hat{W}_{ob}$, then we have the following system of DEs

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 - m\varepsilon_{o\xi} \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(2)} +$$
$$+ \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & 1 + m(1 - \varepsilon_{o\xi}) \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix} = \lambda \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}; \quad \lambda = \rho_a A \frac{R^3}{I_{e\eta}} \alpha^2\,, \tag{14}$$

where λ and α denote the eigenvalues and eigenfrequencies.

For an unloaded beam ($\varepsilon_{o\xi} = 0$), we get back the equations, which govern the free vibrations – compare equation

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(4)} + \begin{bmatrix} -m & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(2)} + \begin{bmatrix} 0 & -m \\ m & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}^{(1)} + \begin{bmatrix} 0 & 0 \\ 0 & m+1 \end{bmatrix} \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix} = \lambda \begin{bmatrix} \hat{U}_{ob} \\ \hat{W}_{ob} \end{bmatrix}$$

to equation (11) in [9]. This system is associated with the boundary conditions valid for pinned-fixed beams and together they constitute an eigenvalue problem. The left side of equation (14) can be rewritten in the brief form

$$\mathbf{K}[\mathbf{y}(\varphi), \varepsilon_{o\xi}] = \overset{4}{\mathbf{P}}\mathbf{y}^{(4)} + \overset{2}{\mathbf{P}}\mathbf{y}^{(2)} + \overset{1}{\mathbf{P}}\mathbf{y}^{(1)} + \overset{0}{\mathbf{P}}\mathbf{y}^{(0)}, \qquad \mathbf{y}^T = \left[\hat{U}_{ob} \mid \hat{W}_{ob}\right]. \tag{15}$$

Observe that the i-th eigenfrequency α_i in the eigenvalue problem depends on the heterogeneity parameters m and ρ_a; and also, on the magnitude and the direction of the concentrated force. The effects of the latter one are accounted through the axial strain: $\varepsilon_{o\xi} = \varepsilon_{o\xi}(\mathcal{P})$. Here $\mathcal{P}$ is a dimensionless load, defined by $\mathcal{P} = P_\zeta R^2 \vartheta/(2I_{e\eta})$.

3. THE GREEN FUNCTION MATRIX

Differential equations (15) are degenerated, since the matrix $\overset{4}{\mathbf{P}}$ has no inverse. Let $\mathbf{r}(\varphi)$ be a prescribed inhomogeneity. Consider the boundary value problems defined by

$$\mathbf{K}(\mathbf{y}) = \sum_{\nu=0}^{4} \overset{\nu}{\mathbf{P}}(\varphi)\mathbf{y}^{(\nu)}(\varphi) = \mathbf{r}(\varphi)\,, \qquad \overset{3}{\mathbf{P}}(\varphi) = \mathbf{0} \tag{16}$$

and the boundary conditions valid for pinned-fixed beams:

$$\hat{U}_{ob}(-\vartheta) = 0 \quad \hat{W}_{ob}(-\vartheta) = 0 \quad \hat{W}^{(2)}_{ob}(-\vartheta) = 0 \quad | \quad \hat{U}_{ob}(\vartheta) = 0 \quad \hat{W}_{ob}(\vartheta) = 0 \quad \hat{W}^{(1)}_{ob}(\vartheta) = 0\,. \tag{17}$$

Solution to the homogeneous part of differential equation (15) depends on whether the axial strain $\varepsilon_{o\xi}$ is positive or negative – i.e.: whether the concentrated force is directed upwards or downwards. Let

$$\chi^2 = \begin{cases} 1 - m\varepsilon_{o\xi} & \text{if} \quad \varepsilon_{o\xi} < 0 \\ m\varepsilon_{o\xi} - 1 & \quad\;\; \varepsilon_{o\xi} > 0 \text{ and } m\varepsilon_{o\xi} > 1\,. \end{cases} \tag{18}$$

This solution is of the form

$$\mathbf{y} = \left[\sum_{i=1}^{4} \underset{(2\times2)}{\mathbf{Y}}{}_i \underset{(2\times2)}{\mathbf{C}}{}_i\right] \underset{(2\times1)}{\mathbf{e}}\,, \quad \text{where} \tag{19a}$$

$$\mathbf{Y}_1 = \begin{bmatrix} \cos\varphi & 0 \\ \sin\varphi & 0 \end{bmatrix}, \ \mathbf{Y}_2 = \begin{bmatrix} -\sin\varphi & 0 \\ \cos\varphi & 0 \end{bmatrix}, \ \mathbf{Y}_3 = \begin{bmatrix} \cos\chi\varphi & \mathcal{M}\varphi \\ \chi\sin\chi\varphi & -1 \end{bmatrix}, \ \mathbf{Y}_4 = \begin{bmatrix} -\sin\chi\varphi & 1 \\ \chi\cos\chi\varphi & 0 \end{bmatrix} \tag{19b}$$

when $\varepsilon_{o\xi} < 0$. However, $\mathbf{Y}_3$ and $\mathbf{Y}_4$ are different when $\varepsilon_{o\xi} > 0$ and $m\varepsilon_{o\xi} > 1$:

$$\mathbf{Y}_3 = \begin{bmatrix} \cosh\chi\varphi & \mathcal{M}\varphi \\ \chi\sinh\chi\varphi & -1 \end{bmatrix}, \ \mathbf{Y}_4 = \begin{bmatrix} -\sinh\chi\varphi & 1 \\ \chi\cosh\chi\varphi & 0 \end{bmatrix}. \tag{19c}$$

In the above relations $\mathbf{C}_i$ are arbitrary constant matrices, $\mathbf{e}$ are arbitrary column matrices and $\mathcal{M} = (m+1)/[m\,(1+\varepsilon_{o\xi})]$. Solutions to the boundary value problems (16) and (17) are sought in the form

$$\mathbf{y}(\varphi) = \int_a^b \mathbf{G}(\varphi,\psi)\mathbf{r}(\psi)\mathrm{d}\psi\,, \quad \mathbf{G}(\varphi,\psi) = \begin{bmatrix} G_{11}(\varphi,\psi) & G_{12}(\varphi,\psi) \\ G_{21}(\varphi,\psi) & G_{22}(\varphi,\psi) \end{bmatrix}, \tag{20}$$

where $\mathbf{G}(\varphi,\psi)$ is the Green function matrix, defined by the following properties [5]:

(1) The Green function matrix is continuous in φ and ψ in each of the triangular ranges $-\vartheta \le \varphi \le \psi \le \vartheta$ and $-\vartheta \le \xi \le \varphi \le \vartheta$. The elements $(G_{11}(\varphi,\psi), G_{12}(\varphi,\psi))$ $[G_{21}(\varphi,\psi), G_{22}(\varphi,\psi)]$ are (2 times) [4 times] differentiable with respect to φ, and the following derivatives are continuous in φ and ψ:

$$\frac{\partial^\nu \mathbf{G}(\varphi,\psi)}{\partial x^\nu} = \mathbf{G}^{(\nu)}(\varphi,\psi)\,, \quad \nu = 1,2\,, \quad \frac{\partial^\nu G_{2i}(\varphi,\psi)}{\partial x^\nu} = G_{2i}^{(\nu)}(\varphi,\psi)\,, \quad \nu = 1,\dots,4;\ i = 1,2. \tag{21}$$

(2) Let ψ be fixed in $[-\vartheta,\vartheta]$. Although the functions $G_{11}(\varphi,\psi)\,, G_{12}^{(1)}(\varphi,\psi)\,, G_{21}^{(\nu)}(\varphi,\psi)$ $\nu = 1,2,3\,; G_{22}^{(\nu)}(\varphi,\psi)$ $\nu = 1,2$ are continuous everywhere, the derivatives $G_{11}^{(1)}(\varphi,\psi)$,$G_{22}^{(3)}(\varphi,\psi)$ have a jump at $\varphi = \psi$:

$$\lim_{\varepsilon\to 0}\left[G_{11}^{(1)}(\varphi+\varepsilon,\varphi) - G_{11}^{(1)}(\varphi-\varepsilon,\varphi)\right] = 1/\overset{1}{P}_{11}(\varphi), \ \lim_{\varepsilon\to 0}\left[G_{22}^{(3)}(\varphi+\varepsilon,\varphi) - G_{22}^{(3)}(\varphi-\varepsilon,\varphi)\right] = 1/\overset{4}{P}_{22}(\varphi)\,. \tag{22}$$

(3) Let $\boldsymbol{\alpha}$ denote an arbitrary constant vector. For a given $\psi \in [-\vartheta,\vartheta]$, the vector $\mathbf{G}(\varphi,\psi)\boldsymbol{\alpha}$, as a function of φ $(\varphi \neq \psi)$ should satisfy the homogeneous differential equation $\mathbf{K}\left[\mathbf{G}(\varphi,\psi)\boldsymbol{\alpha}\right] = \mathbf{0}$.

(4) The vector $\mathbf{G}(\varphi,\psi)\boldsymbol{\alpha}$, as a function of φ, should satisfy the boundary conditions (17). Moreover, there exists only one Green function matrix to each of the boundary value problems.

If the Green function matrix exists then (20) satisfies the differential equations (16) and the boundary conditions (17).

Consider the system of differential equations in the form of

$$\mathbf{K}[\mathbf{y}] = \lambda\mathbf{y}\,, \tag{23}$$

where $\mathbf{K}[\mathbf{y}]$ is given by (15) and λ is the unknown eigenvalue. The system of ordinary DEs (23) and the homogeneous boundary conditions (17) constitute a boundary value problem, which is, in fact, an eigenvalue problem with λ as the eigenvalue.

Vectors $\mathbf{u}^T = [u_1|u_2]$ and $\mathbf{v}^T = [v_1|v_2]$ are comparison vectors, if they are different from zero, satisfy the boundary conditions and are differentiable as many times as required. The eigenvalue problems (23), (17) are self-adjoint if the product

$$(\mathbf{u},\mathbf{v})_M = \int_{-\vartheta}^{\vartheta} \mathbf{u}^T\mathbf{K}\mathbf{v}\,\mathrm{d}\varphi \tag{24}$$

is commutative, i.e., $(\mathbf{u},\mathbf{v})_M = (\mathbf{v},\mathbf{u})_M$ over the set of comparison vectors and it is positive definite for any comparison vector $\mathbf{u}$, if $(\mathbf{u},\mathbf{u})_M > 0$. If the eigenvalue problems (23), (17) are self-adjoint, then the related Green function matrices are cross-symmetric: $\mathbf{G}(\varphi,\psi) = \mathbf{G}^T(\psi,\varphi)$.

4. Numerical solution to the eigenvalue problems

Making use of (20), the eigenvalue problems (23), (17) can be replaced by homogeneous integral equation systems:

$$\mathbf{y}(\varphi) = \lambda\int_{-\vartheta}^{\vartheta} \mathbf{G}(\varphi,\psi)\mathbf{y}(\psi)\mathrm{d}\psi\,. \tag{25}$$

Numerical solution to such problems can be sought e.g., by quadrature methods [10]. Consider the integral formula

$$J(\phi) = \int_{-\vartheta}^{\vartheta} \phi(\psi)\,\mathrm{d}\psi \equiv \sum_{j=0}^{n} w_j\phi(\psi_j), \qquad \psi_j \in [-\vartheta,\vartheta]\,, \tag{26}$$

where $\psi_j(\varphi)$ is a vector and w_j are the known weights. Having utilized the latter equation, we obtain from (25) that

$$\sum_{j=0}^{n} w_j\mathbf{G}(\varphi,\psi_j)\tilde{\mathbf{y}}(\psi_j) = \tilde{\kappa}\tilde{\mathbf{y}}(\varphi) \qquad \tilde{\kappa} = 1/\tilde{\lambda} \qquad \in [-\vartheta,\vartheta] \tag{27}$$

is the solution, which yields an approximate eigenvalue $\tilde{\lambda} = 1/\tilde{\kappa}$ and the corresponding approximate eigenfunction $\tilde{\mathbf{y}}(\varphi)$. After setting φ to ψ_i $(i = 0, 1, 2, \ldots, n)$, we have

$$\sum_{j=0}^{n} w_j \mathbf{G}(\psi_i, \psi_j)\tilde{\mathbf{y}}(\psi_j) = \tilde{\kappa}\tilde{\mathbf{y}}(\psi_i) \qquad \tilde{\kappa} = 1/\tilde{\lambda} \qquad \psi_i, \psi_j \in [-\vartheta, \vartheta] \,, \quad \text{or} \quad \mathcal{G}\mathcal{D}\tilde{\mathcal{Y}} = \tilde{\kappa}\tilde{\mathcal{Y}} \,, \tag{28}$$

where $\mathcal{G} = [\mathbf{G}(\psi_i, \psi_j)]$ for self-adjoint problems, while $\mathcal{D} = \text{diag}(w_0, \ldots, w_0|\ldots|w_n, \ldots, w_n)$ and $\tilde{\mathcal{Y}}^T = [\tilde{\mathbf{y}}^T(\psi_0)|\tilde{\mathbf{y}}^T(\psi_1)|\ldots|\tilde{\mathbf{y}}^T(\psi_n)]$. After solving the generalized algebraic eigenvalue problem (28), we have the approximate eigenvalues $\tilde{\lambda}_r$ and eigenvectors $\mathcal{Y}_r$, while the corresponding eigenfunction is obtained via substituting into (27):

$$\tilde{\mathbf{y}}_r(\varphi) = \tilde{\lambda}_r \sum_{j=0}^{n} w_j \mathbf{G}(\varphi, \psi_j)\tilde{\mathbf{y}}_r(\psi_j) \qquad r = 0, 1, 2, \ldots, n \,. \tag{29}$$

Divide the interval $[-\vartheta, \vartheta]$ into equidistant subintervals of length h and apply the integration formula to each subinterval. By repeating the line of thought leading to (29), the algebraic eigenvalue problem obtained has the same structure as (29).

It is also possible to consider the integral equations (25) as if they were boundary integral equations and apply isoparametric approximation on the subintervals (elements). If this is the case, one can approximate the eigenfunction on the e-th element (the e-th subinterval which is mapped onto the interval $\gamma \in [-1, 1]$ and is denoted by $\mathfrak{L}_e$) by

$$\overset{e}{\mathbf{y}} = \mathbf{N}_1(\gamma)\overset{e}{\mathbf{y}}_1 + \mathbf{N}_2(\gamma)\overset{e}{\mathbf{y}}_2 + \mathbf{N}_3(\gamma)\overset{e}{\mathbf{y}}_3 \,, \tag{30}$$

where quadratic local approximation is assumed: $\mathbf{N}_i = \text{diag}(N_i),\ N_1 = 0.5\gamma(\gamma - 1),\ N_2 = 1 - \gamma^2,\ N_3 = 0.5\gamma(\gamma + 1),\ \overset{e}{\mathbf{y}}_i$ is the value of the eigenfunction $\mathbf{y}(\varphi)$ at the left endpoint, the midpoint and the right endpoint of the element, respectively. Upon substitution of approximation (30) into (25), we have

$$\tilde{\mathbf{y}}(\varphi) = \tilde{\lambda} \sum_{e=1}^{n_{be}} \int_{\mathfrak{L}_e} \mathbf{G}(x, \gamma)[\mathbf{N}_1(\eta)|\mathbf{N}_2(\gamma)|\mathbf{N}_3(\gamma)]d\gamma \left[\overset{e}{\mathbf{y}}_1|\overset{e}{\mathbf{y}}_2|\overset{e}{\mathbf{y}}_3\right]^T \,, \tag{31}$$

in which, n_{be} is the number of elements. Using equation (31) as a point of departure, and repeating the line of thought leading to (28), we again get an algebraic eigenvalue problem.

5. Computation of the Green function matrices

Based on the definition presented in Section 3, here we show the calculation of the corresponding Green function matrices for the two loading cases of pinned-fixed beams. With regard to property 3, the Green function can be given in the form

$$\underbrace{\mathbf{G}(\varphi, \psi)}_{(2\times 2)} = \sum_{j=1}^{4} \mathbf{Y}_j(\varphi)\left[\mathbf{A}_j(\psi) \pm \mathbf{B}_j(\psi)\right] \,, \tag{32}$$

where (a) the sign is [positive](negative) if $[\varphi \le \psi](\varphi \ge \psi)$; (b) the matrices $\mathbf{A}_j$ and $\mathbf{B}_j$ have the following structure

$$\mathbf{A}_j = \begin{bmatrix} \overset{j}{A}_{11} & \overset{j}{A}_{12} \\ \overset{j}{A}_{21} & \overset{j}{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{j1} & \mathbf{A}_{j2} \end{bmatrix} \,, \qquad \mathbf{B}_j = \begin{bmatrix} \overset{j}{B}_{11} & \overset{j}{B}_{12} \\ \overset{j}{B}_{21} & \overset{j}{B}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{j1} & \mathbf{B}_{j2} \end{bmatrix} \quad j = 1, \ldots, 4; \tag{33}$$

(c) the coefficients in $\mathbf{B}_j$ are independent of the boundary conditions. As the matrices $\mathbf{Y}_3$ and $\mathbf{Y}_4$ are different for $\varepsilon_{o\xi} < 0$ and for $\varepsilon_{o\xi} > 0$, when $m\varepsilon_{o\xi} > 1$, we deal with the two possibilities separately.

The Green functions matrix if $\varepsilon_{o\xi} < 0$. Let us now introduce the following notational conventions

$$a = \overset{1}{B}_{1i},\ b = \overset{2}{B}_{1i},\ c = \overset{3}{B}_{1i},\ d = \overset{3}{B}_{2i},\ e = \overset{4}{B}_{1i},\ f = \overset{4}{B}_{2i} \,. \tag{34}$$

We note that $\overset{1}{B}_{21} = \overset{2}{B}_{21} = \overset{1}{B}_{22} = \overset{2}{B}_{22} = 0$ – see Section 3. The systems of equations for the unknowns $a, \ldots, f$ can be set up by fulfilling the continuity and discontinuity conditions mentioned in property 1 and 2 for the Green function matrix if $\varphi = \psi$. Therefore, if $i = 1$, we have

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cos(\chi\psi) & \mathcal{M}\psi & -\sin(\chi\psi) & 1 \\ \sin\psi & \cos\psi & \chi\sin(\chi\psi) & -1 & \chi\cos(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi\sin(\chi\psi) & \mathcal{M} & -\chi\cos(\chi\psi) & 0 \\ \cos\psi & -\sin\psi & \chi^2\cos(\chi\psi) & 0 & -\chi^2\sin(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi^3\sin(\chi\psi) & 0 & -\chi^3\cos(\chi\psi) & 0 \\ -\cos\psi & \sin\psi & -\chi^4\cos(\chi\psi) & 0 & \chi^4\sin(\chi\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2m} \\ 0 \\ 0 \\ 0 \end{bmatrix} \,, \tag{35}$$

from where we get the constants as

$$
\begin{aligned}
a &= \overset{1}{B}_{11} = \frac{\chi^2}{(1-\chi^2)(1-\mathcal{M})\,m}\frac{\sin\psi}{2}\,, & b &= \overset{2}{B}_{11} = \frac{\chi^2}{(1-\chi^2)(1-\mathcal{M})\,m}\frac{\cos\psi}{2}\,,\\
c &= \overset{3}{B}_{11} = -\frac{\chi^2}{(1-\chi^2)(1-\mathcal{M})\,m}\frac{\sin\chi\psi}{2\chi^3}\,, & d &= \overset{3}{B}_{21} = -\frac{1}{2\,(1-\mathcal{M})\,m}\,,\\
e &= \overset{4}{B}_{11} = -\frac{1}{\chi\,(1-\chi^2)(1-\mathcal{M})\,m}\frac{\cos\chi\psi}{2}\,, & f &= \overset{4}{B}_{21} = \frac{1}{2}\mathcal{M}\frac{\psi}{m\,(1-\mathcal{M})}\,.
\end{aligned}
\tag{36}
$$

If $i=2$, then

$$
\begin{bmatrix}
\cos\psi & -\sin\psi & \cos(\chi\psi) & \mathcal{M}\psi & -\sin(\chi\psi) & 1\\
\sin\psi & \cos\psi & \chi\sin(\chi\psi) & -1 & \chi\cos(\chi\psi) & 0\\
-\sin\psi & -\cos\psi & -\chi\sin(\chi\psi) & \mathcal{M} & -\chi\cos(\chi\psi) & 0\\
\cos\psi & -\sin\psi & \chi^2\cos(\chi\psi) & 0 & -\chi^2\sin(\chi\psi) & 0\\
-\sin\psi & -\cos\psi & -\chi^3\sin(\chi\psi) & 0 & -\chi^3\cos(\chi\psi) & 0\\
-\cos\psi & \sin\psi & -\chi^4\cos(\chi\psi) & 0 & \chi^4\sin(\chi\psi) & 0
\end{bmatrix}
\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ -\frac{1}{2} \end{bmatrix}
\tag{37}
$$

is the equation system, the solution of which assumes the form

$$
\begin{aligned}
a &= \overset{1}{B}_{12} = \frac{1}{2}\frac{\cos\psi}{(1-\chi^2)}\,, & b &= \overset{2}{B}_{12} = -\frac{1}{2}\frac{\sin\psi}{(1-\chi^2)}\,, & c &= \overset{3}{B}_{12} = -\frac{1}{2}\frac{\cos\chi\psi}{(1-\chi^2)\,\chi^2}\,,\\
d &= \overset{3}{B}_{22} = 0\,, & e &= \overset{4}{B}_{12} = \frac{1}{2}\frac{\sin\chi\psi}{(1-\chi^2)\,\chi^2}\,, & f &= \overset{4}{B}_{22} = \frac{1}{2\chi^2}\,.
\end{aligned}
\tag{38}
$$

Regarding the unknown scalars $\overset{1}{A}_{1i}(\psi)$, $\overset{2}{A}_{1i}(\psi)$, $\overset{3}{A}_{1i}(\psi)$, $\overset{3}{A}_{2i}(\psi)$, $\overset{4}{A}_{1i}(\psi)$, $\overset{4}{A}_{2i}(\psi)\,, i=1,2; \psi\in[-\vartheta,\vartheta]$ in the matrices $\mathbf{A}_j$, property (4), that is the boundary conditions (17) yield

$$
\begin{bmatrix}
\cos\vartheta & \sin\vartheta & \cos(\chi\vartheta) & -\mathcal{M}\vartheta & \sin(\chi\vartheta) & 1\\
\cos\vartheta & -\sin\vartheta & \cos(\chi\vartheta) & \mathcal{M}\vartheta & -\sin(\chi\vartheta) & 1\\
-\sin\vartheta & \cos\vartheta & -\chi\sin(\chi\vartheta) & -1 & \chi\cos(\chi\vartheta) & 0\\
\sin\vartheta & \cos\vartheta & \chi\sin(\chi\vartheta) & -1 & \chi\cos(\chi\vartheta) & 0\\
\cos\vartheta & \sin\vartheta & \chi^2\cos(\chi\vartheta) & 0 & \chi^2\sin(\chi\vartheta) & 0\\
-\sin\vartheta & -\cos\vartheta & -\chi^3\sin(\chi\vartheta) & 0 & -\chi^3\cos(\chi\vartheta) & 0
\end{bmatrix}
\begin{bmatrix} \overset{1}{A}_{1i}\\ \overset{2}{A}_{1i}\\ \overset{3}{A}_{1i}\\ \overset{3}{A}_{2i}\\ \overset{4}{A}_{1i}\\ \overset{4}{A}_{2i} \end{bmatrix}
=
$$

$$
=\begin{bmatrix}
-a\cos\vartheta - b\sin\vartheta - c\cos(\chi\vartheta) + d\mathcal{M}\vartheta - e\sin(\chi\vartheta) - f\\
a\cos\vartheta - b\sin\vartheta + c\cos(\chi\vartheta) + d\mathcal{M}\vartheta - e\sin(\chi\vartheta) + f\\
a\sin\vartheta - b\cos\vartheta + c\chi\sin(\chi\vartheta) + d - e\chi\cos(\chi\vartheta)\\
a\sin\vartheta + b\cos\vartheta + c\chi\sin(\chi\vartheta) - d + e\chi\cos(\chi\vartheta)\\
-a\cos\vartheta - b\sin\vartheta - c\chi^2\cos(\chi\vartheta) - e\chi^2\sin(\chi\vartheta)\\
-a\sin\vartheta - b\cos\vartheta - c\chi^3\sin(\chi\vartheta) - e\chi^3\cos(\chi\vartheta)
\end{bmatrix}.
\tag{39}
$$

Closed form solution to this system was generated using Maple 15. The unknowns are not detailed here.

Calculation of the Green functions matrix if $\varepsilon_{o\xi}>0$ and $m\varepsilon_{o\xi}>1$. For the other loading case, repeating a procedure similar to the procedure leading to (35), we get the following equations if $i=1$

$$
\begin{bmatrix}
\cos\psi & -\sin\psi & \cosh(\chi\psi) & \mathcal{M}\psi & \sinh(\chi\psi) & 1\\
\sin\psi & \cos\psi & -\chi\sinh(\chi\psi) & -1 & -\chi\cosh(\chi\psi) & 0\\
-\sin\psi & -\cos\psi & \chi\sinh(\chi\psi) & \mathcal{M} & \chi\cosh(\chi\psi) & 0\\
\cos\psi & -\sin\psi & -\chi^2\cosh(\chi\psi) & 0 & -\chi^2\sinh(\chi\psi) & 0\\
-\sin\psi & -\cos\psi & -\chi^3\sinh(\chi\psi) & 0 & -\chi^3\cosh(\chi\psi) & 0\\
-\cos\psi & \sin\psi & -\chi^4\cosh(\chi\psi) & 0 & -\chi^4\sinh(\chi\psi) & 0
\end{bmatrix}
\begin{bmatrix} a\\ b\\ c\\ d\\ e\\ f \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ \frac{1}{2m}\\ 0\\ 0\\ 0 \end{bmatrix}.
\tag{40}
$$

The solutions are as follows

$$
\begin{aligned}
a &= \overset{1}{B}_{11} = -\frac{\chi^2}{(1+\chi^2)(1-\mathcal{M})\,m}\frac{\sin\psi}{2}\,, & b &= \overset{2}{B}_{11} = -\frac{\chi^2}{(1+\chi^2)(1-\mathcal{M})\,m}\frac{\cos\psi}{2}\,,\\
c &= \overset{3}{B}_{11} = -\frac{1}{\chi\,(1+\chi^2)(1-\mathcal{M})\,m}\frac{\sinh\chi\psi}{2}\,, & d &= \overset{3}{B}_{21} = -\frac{1}{2\,(1-\mathcal{M})\,m}\,,\\
e &= \overset{4}{B}_{11} = \frac{1}{\chi\,(1+\chi^2)(1-\mathcal{M})\,m}\frac{\cosh\chi\psi}{2}\,, & f &= \overset{4}{B}_{21} = \frac{1}{2\,(1-\mathcal{M})\,m}\mathcal{M}\psi\,.
\end{aligned}
\tag{41}
$$

If $i = 2$

$$\begin{bmatrix} \cos\psi & -\sin\psi & \cosh(\chi\psi) & \mathcal{M}\psi & \sinh(\chi\psi) & 1 \\ \sin\psi & \cos\psi & -\chi\sinh(\chi\psi) & -1 & -\chi\cosh(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & \chi\sinh(\chi\psi) & \mathcal{M} & \chi\cosh(\chi\psi) & 0 \\ \cos\psi & -\sin\psi & -\chi^2\cosh(\chi\psi) & 0 & -\chi^2\sinh(\chi\psi) & 0 \\ -\sin\psi & -\cos\psi & -\chi^3\sinh(\chi\psi) & 0 & -\chi^3\cosh(\chi\psi) & 0 \\ -\cos\psi & \sin\psi & -\chi^4\cosh(\chi\psi) & 0 & -\chi^4\sinh(\chi\psi) & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \tag{42}$$

is the equation system to be solved – compare it to (37) – and the solutions we have obtained are

$$\begin{aligned} a &= \overset{1}{B}_{12} = \frac{1}{2}\frac{\cos\psi}{(1+\chi^2)}\,, \quad b = \overset{2}{B}_{12} = -\frac{1}{2}\frac{\sin\psi}{(1+\chi^2)}\,, \qquad c = \overset{3}{B}_{12} = \frac{1}{2}\frac{\cosh\chi\psi}{(1+\chi^2)\,\chi^2} \\ d &= \overset{3}{B}_{22} = 0\,, \qquad\qquad e = \overset{4}{B}_{12} = -\frac{1}{2}\frac{\sinh\chi\psi}{\chi^2\,(1+\chi^2)}\,, \quad f = \overset{4}{B}_{22} = -\frac{1}{2\chi^2}\,. \end{aligned} \tag{43}$$

For the elements of the matrices $\mathbf{A}_j$, boundary conditions (17) yield the equation system

$$\begin{bmatrix} \cos\vartheta & \sin\vartheta & \cosh(\chi\vartheta) & -\mathcal{M}\vartheta & -\sinh(\chi\vartheta) & 1 \\ \cos\vartheta & -\sin\vartheta & \cosh(\chi\vartheta) & \mathcal{M}\vartheta & \sinh(\chi\vartheta) & 1 \\ -\sin\vartheta & \cos\vartheta & \chi\sinh(\chi\vartheta) & -1 & -\chi\cosh(\chi\vartheta) & 0 \\ \sin\vartheta & \cos\vartheta & -\chi\sinh(\chi\vartheta) & -1 & -\chi\cosh(\chi\vartheta) & 0 \\ \cos\vartheta & \sin\vartheta & -\chi^2\cosh(\chi\vartheta) & 0 & \chi^2\sinh(\chi\vartheta) & 0 \\ -\sin\vartheta & -\cos\vartheta & -\chi^3\sinh(\chi\vartheta) & 0 & -\chi^3\cosh(\chi\vartheta) & 0 \end{bmatrix} \begin{bmatrix} \overset{1}{A}_{1i} \\ \overset{2}{A}_{1i} \\ \overset{3}{A}_{1i} \\ \overset{3}{A}_{2i} \\ \overset{4}{A}_{1i} \\ \overset{4}{A}_{2i} \end{bmatrix} =$$

$$= \begin{bmatrix} -a\cos\vartheta - b\sin\vartheta - c\cosh(\chi\vartheta) + d\mathcal{M}\vartheta + e\sinh(\chi\vartheta) - f \\ a\cos\vartheta - b\sin\vartheta + c\cosh(\chi\vartheta) + d\mathcal{M}\vartheta + e\sinh(\chi\vartheta) + f \\ a\sin\vartheta - b\cos\vartheta - c\chi\sinh(\chi\vartheta) + d + e\chi\cosh(\chi\vartheta) \\ a\sin\vartheta + b\cos\vartheta - c\chi\sinh(\chi\vartheta) - d - e\chi\cosh(\chi\vartheta) \\ -a\cos\vartheta - b\sin\vartheta + c\chi^2\cosh(\chi\vartheta) - e\chi^2\sinh(\chi\vartheta) \\ -a\sin\vartheta - b\cos\vartheta - c\chi^3\sinh(\chi\vartheta) - e\chi^3\cosh(\chi\vartheta) \end{bmatrix}\,. \tag{44}$$

the solutions of which are omitted here.

6. The load-strain relation and the critical strain

In practise, the loading is generally the known quantity. However, the formulation has the axial strain $\varepsilon_{o\xi}$ as a parameter. For a first, linearized model, the effect the deformations have on the equilibrium is neglected. We can establish the load-strain relationship $\varepsilon_{o\xi} = \varepsilon_{o\xi}(\mathcal{P})$ on the basis of differential equations (8) if $f_t = f_n = 0$, and by applying the

$$U_o|_{\pm\vartheta} = W_o|_{\pm\vartheta} = M|_{-\vartheta} = \psi_{o\eta}|_{+\vartheta} = 0\,, \tag{45a}$$

$$\begin{aligned} &U_o|_{\varphi=-0} = U_o|_{\varphi=+0}\,, \qquad W_o|_{\varphi=-0} = W_o|_{\varphi=+0}\,, \qquad \psi_{o\eta}|_{\varphi=-0} = \psi_{o\eta}|_{\varphi=+0}\,, \\ &N|_{\varphi=-0} = N|_{\varphi=+0}\,, \qquad M|_{\varphi=-0} = M|_{\varphi=+0}\,, \qquad \mathrm{d}M/\mathrm{d}s|_{\varphi=+0} - \mathrm{d}M\mathrm{d}s|_{\varphi=-0} - P_\zeta = 0 \end{aligned} \tag{45b}$$

boundary and continuity (discontinuity) conditions prescribed at the crown point. Here the physical quantities can all be given in terms of the dimensionless displacements U_o and W_o as

$$\psi_{o\eta} = U_o - W_o^{(1)}\,, \quad N = A_e\varepsilon_{o\xi} - \frac{M}{\rho_o} \approx A_e\varepsilon_{o\xi}\,, \quad M = -\frac{I_{e\eta}}{\rho_o^2}\left(W_o^{(2)} + W_o\right)\,. \tag{46}$$

After solving the boundary value problem defined by the ODEs (8) ($f_t = f_n = 0$) and the above continuity and discontinuity conditions we get the axial strain in the following form

$$\begin{aligned} \varepsilon(\mathcal{P}, m, \vartheta) = &-\frac{\mathcal{P}}{\vartheta}\left[(3\vartheta\cos\vartheta - 4\cos\vartheta\sin\vartheta + 4\sin\vartheta)\cos^2\vartheta + \left(\vartheta^2 - 2 + 2\cos\vartheta\right)\sin\vartheta + \vartheta\,(2 - 5\cos\vartheta)\right] \times \\ &\times \frac{1}{m\left[(-2\vartheta\sin\vartheta + 11\cos\vartheta - 4\vartheta\cos^2\vartheta\sin\vartheta - 7\cos^3\vartheta)\cos\vartheta - 4 + 3\vartheta^2\right] + 2\vartheta\,(\cos\vartheta\sin\vartheta - 2\cos^3\vartheta\sin\vartheta + \vartheta)}\,. \end{aligned} \tag{47}$$

If $\mathcal{P}$ is [negative] (positive), then $\varepsilon_{o\xi}$ is [negative] (positive). The critical strain, at which, curved beams lose their stability, can be obtained by solving the eigenvalue problem governed by equations (13) – with the right side set to zero – and the corresponding boundary conditions. The eigenvalue sought is $\chi = \sqrt{1 - m\varepsilon_{o\xi}}$.

The general solution for the displacement increments and the boundary conditions to be satisfied are:

$$W_{ob} = -E_2 - E_3\cos\varphi + E_4\sin\varphi - \chi E_5\cos\chi\varphi + \chi E_6\sin\chi\varphi\,, \tag{48}$$

$$U_{ob} = E_1 + E_2\mathcal{M}\varphi + E_3\sin\varphi + E_4\cos\varphi + E_5\sin\chi\varphi + E_6\cos\chi\varphi\,, \tag{49}$$

$$U_{ob}\big|_{\pm\vartheta} = W_{ob}\big|_{\pm\vartheta} = W_{ob}^{(2)}\Big|_{-\vartheta} = W_{ob}^{(1)}\Big|_{+\vartheta} = 0\,. \tag{50}$$

Here E_i $(i = 1, \dots, 6)$ are undetermined constants of integration. The boundary conditions lead to the homogenous equation system

$$\begin{bmatrix} 1 & -\mathcal{M}\vartheta & -\sin\vartheta & \cos\vartheta & -\sin\chi\vartheta & \cos\chi\vartheta \\ 1 & \mathcal{M}\vartheta & \sin\vartheta & \cos\vartheta & \sin\chi\vartheta & \cos\chi\vartheta \\ 0 & 1 & \cos\vartheta & \sin\vartheta & \chi\cos\chi\vartheta & \chi\sin\chi\vartheta \\ 0 & 1 & \cos\vartheta & -\sin\vartheta & \chi\cos\chi\vartheta & -\chi\sin\chi\vartheta \\ 0 & 0 & \cos\vartheta & \sin\vartheta & \chi^3\cos\chi\vartheta & \chi^3\sin\chi\vartheta \\ 0 & 0 & \sin\vartheta & \cos\vartheta & \chi^2\sin\chi\vartheta & \chi^2\cos\chi\vartheta \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{51}$$

The vanishing of the determinant results in the following non-linear equation:

$$\begin{aligned} &\left[\chi^4\left(1-2\cos^2\chi\vartheta\right)+\left(1-2\chi^2\right)\sin^2\chi\vartheta+\chi\vartheta\left(\chi^2-1\right)\cos\chi\vartheta\sin\chi\vartheta\right]\sin^2\vartheta+\chi^2\vartheta\left(\chi^2-1\right)\cos\vartheta\sin\vartheta\cos^2\chi\vartheta = \\ &=\left[\left(\chi\vartheta\left(\chi^2-1\right)\cos\chi\vartheta+\sin\chi\vartheta\right)\cos\vartheta\;+\chi^2\vartheta\left(\chi^2-1\right)\sin\vartheta\sin\chi\vartheta+\chi\left(-\chi^2-1\right)\sin\vartheta\cos\chi\vartheta\right]\cos\vartheta\sin\chi\vartheta\,. \end{aligned} \tag{52}$$

The lowest reasonable critical value is then approximated by the polynomial

$$\chi\vartheta = g_{pf}\left(\vartheta\right) = 0.014\,875,\vartheta^5 - 0.078\,701\,\vartheta^4 + 0.168\,958\,\vartheta^3 - 0.119\,606\,\vartheta^2 + 0.057\,002\,\vartheta + 3.749\,293\,, \tag{53}$$

and the critical strain is

$$\varepsilon_{o\xi\ \mathrm{crit}} = -\frac{1}{m}\left(\chi^2-1\right) = -\frac{1}{m}\left[\left(\frac{g_{pf}}{\vartheta}\right)^2-1\right]. \tag{54}$$

7. Computational results

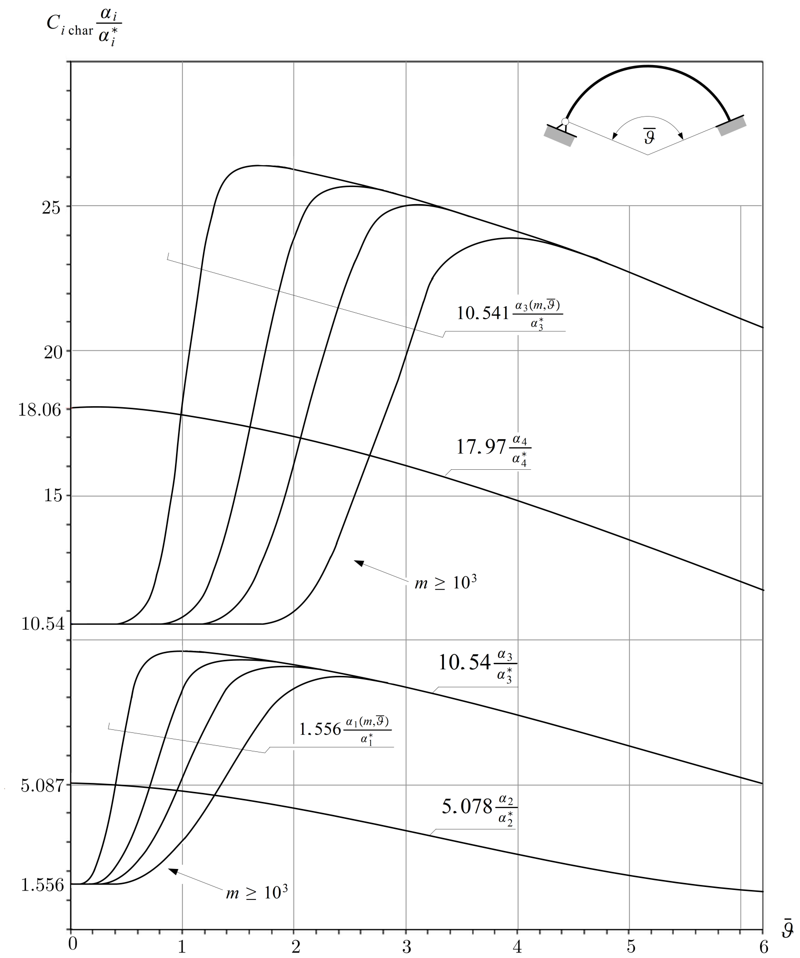

Figure 2. Results for pinned-fixed beams when $\varepsilon_{o\xi} \simeq 0$

We have solved the eigenvalue problems governing the vibrations of curved beams using a Fortran90 code. The numerical results have been compared to those valid for the free vibrations of curved beams with the same geometry and material. For more details about the natural frequencies of planar curved beams see [5].

When we set the strain to a very small value, i.e., to $|\varepsilon_{o\xi}| = |\varepsilon_{o\xi\,\mathrm{crit}} \cdot 10^{-6}|$ for both loading cases, we get back the results valid for the free vibrations of curved beams – see [5] and [11] for details.

It is known that – see e.g.: [9] – the i-th eigenfrequency for the free transverse vibrations of heterogeneous straight beams is obtained from the relation

$$\alpha_i^* = \frac{C_{i,\,\mathrm{char}}\pi^2}{\sqrt{\frac{\rho_a A}{I_{e\eta}}}\ell_b^2}\,, \tag{55}$$

where the constant $C_{i,\,\mathrm{char}}$ depends on the supports and the ordinal number of the frequency sought. This time $C_{1,\,\mathrm{char}} = 1.556, C_{2,\,\mathrm{char}} = 5.078, C_{3,\,\mathrm{char}} = 10.541, C_{4,\,\mathrm{char}} = 17.97$ and ℓ_b is the length of the beam. If we recall Eq.(14)$_2$ which, for such a small strain considered, expresses the relation between the eigenvalues λ_i and the eigenfrequencies $\alpha_i = \alpha_{i\,\mathrm{free}}$ for the free vibrations of curved beams we may write

$$C_{i,\mathrm{char}}\frac{\alpha_i}{\alpha_i^*} = \frac{\frac{\sqrt{\lambda_i}}{\sqrt{\frac{\rho_a A}{I_{e\eta}}}R^2}}{\frac{\pi^2}{\sqrt{\frac{\rho_a A}{I_{e\eta}}}\ell_r^2}} = \frac{\bar{\vartheta}^2\sqrt{\lambda_i}}{\pi^2}\,. \tag{56}$$

This is the connection between the natural frequencies of curved and straight beams with the same length ($\ell_b = R\bar{\vartheta}$), cross-section and material. In Figure 2, this ratio is plotted against the central angle $\bar{\vartheta}$ of the curved beam. Four different values of the parameter m were picked: $(1, 3.4, 12, 100) \cdot 10^3$. Observe that the ratio of the even natural frequencies are independent of m, while the odd ones are not. It is also important to mention that the frequency spectrum changes as $\bar{\vartheta}$ increases – e.g.: the first eigenfrequency becomes the second one in terms of its magnitude if $\bar{\vartheta}$ is sufficiently great.

When dealing with the free longitudinal vibrations of fixed-fixed rods, the natural frequencies assume the form [11]

$$\hat{\alpha}_i = \frac{K_{i\,\mathrm{char}}}{\ell_r}\sqrt{\frac{E}{\rho_a}}\pi\,, \tag{57}$$

where the constant $K_{i\,\mathrm{char}} = i;\ (i = 1, 2, 3, \ldots)$ and ℓ_r is the length of the rod. If we recall Eq.(14)$_2$, we can compare the eigenfrequencies of curved beams (given that $|\varepsilon_{o\xi}| = |\varepsilon_{o\xi\,\mathrm{crit}} \cdot 10^{-6}| \simeq 0$ when calculating λ_i) to those of rods by

$$K_{i\,\mathrm{char}}\frac{\alpha_i}{\hat{\alpha}_i} = \frac{1}{\sqrt{m}}\frac{\bar{\vartheta}}{\pi}\sqrt{\lambda_i}\,. \tag{58}$$

These quotients for $i = 1, 2$ are plotted in Figure 3. We found that the ratios do not depend on the parameter m and these are equal to 1 and 2 respectively if the central angle is sufficiently small. We remark that these tendencies are the same with a good accuracy for pinned-pinned and for fixed-fixed curved beams as well.

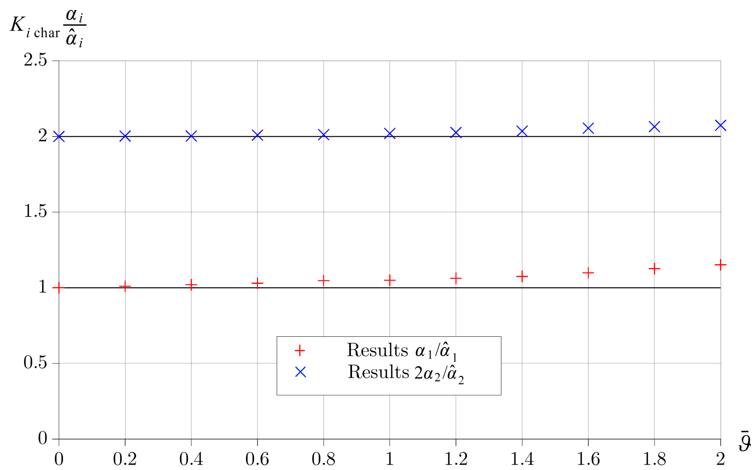

Figure 3. Results for pinned-fixed beams when $\varepsilon_{o\xi} \simeq 0$

In the forthcoming, the effect of the concentrated load to the length of the centerline is taken into account. In what follows, regarding our notations, α_i is i-th eigenfrequency of the loaded curved beam, while the eigenfrequencies that belong to the free vibrations (the beam is unloaded) are denoted by $\alpha_{i\,\mathrm{free}}$.

Figure 4 represents the quotient $\alpha_1^2/\alpha_{1\,\mathrm{free}}^2$ against the quotient $|\varepsilon_{o\xi}/\varepsilon_{o\xi\,\mathrm{crit}}|$ both for a negative and a positive P_ζ. We remark that this time the subscript 1 always refers to the lowest frequencies (which do not coincide with the first one every time – see Figure 2). The frequencies under [compression] <tension> [decrease] <increase> almost linearly and independently of m and ϑ, given that $m >\sim 10\,000$ and $\bar{\vartheta} >\sim 1$. These relationships can be approximated with a very good accuracy by

$$\frac{\alpha_1^2}{\alpha_{1\ \text{free}}^2} = 1.000\,848\,535 - 0.983\,386\,732\frac{|\varepsilon_{o\xi}|}{\varepsilon_{o\xi\ crit}} - 0.174\,018\,254\left(\frac{\varepsilon_{o\xi}}{\varepsilon_{o\xi\ crit}}\right)^2 , \quad \text{if} \quad \varepsilon_{o\xi} < 0\,, \tag{59}$$

$$\frac{\alpha_1^2}{\alpha_{1\ \text{free}}^2} = 1.000\,198\,503 + 0.986\,131\,634\frac{|\varepsilon_{o\xi}|}{\varepsilon_{o\xi\ crit}} - 0.008\,370\,551\left(\frac{\varepsilon_{o\xi}}{\varepsilon_{o\xi\ crit}}\right)^2 , \quad \text{if} \quad \varepsilon_{o\xi} > 0. \tag{60}$$

We again remark that these results are almost the same, when the curved beam is pinned-pinned or fixed-fixed.

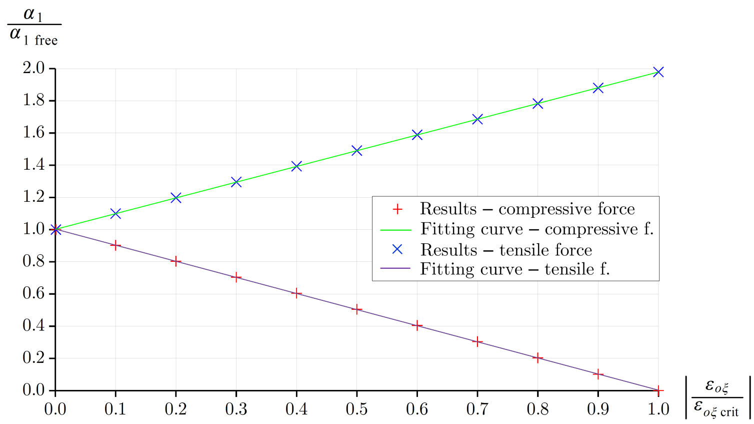

Figure 4. Results for the two loading cases of pinned-fixed beams

8. Concluding remarks

In accordance with our aims, we have investigated the vibrations of curved beams with cross-sectional heterogeneity under a centra load (a vertical force) exerted at the crown point. We have derived the governing equations of the boundary value problems, which make it possible to clarify how the radial load affects the natural frequencies.

For pinned-fixed beams, we have determined the Green function matrices assuming that the beams are prestressed by a radial load. When computing the corresponding matrices, we had to take into account that the system of differential equations that govern the problem are degenerated. Making use of the Green function matrices, we have reduced the self-adjoint eigenvalue problems set up for the eigenfrequencies to eigenvalue problems governed by homogeneous systems of Fredholm integral equations. These integral equations can be used for those loads, which results in a constant axial strain on the E-weighted centerline.

Numerical solutions were provided graphically. For the loaded beams considered, the quotient $\alpha_1^2/\alpha_{1\,\text{free}}^2$ depends linearly with a good accuracy on the axial strain $\varepsilon_{o\xi}$. With the knowledge of the relationship $\varepsilon_{o\xi} = \varepsilon_{o\xi}(\mathcal{P})$, we can determine the strain, which belongs to a given load and then the natural frequencies of the loaded structure.

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