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LARGE DEFORMATIONS AT THIN PLANE CIRCULAR PLATES BENDING UNDER CONSTANT LOADING, TAKING INTO ACCOUNT THE MEMBRANE STRESSES PART II – BENDING UNDER INITIAL LOADING OF THE PLATES MEAN PLANE

Gheorghe N. RADU*
*TRANSILVANIA University of Brasov, Dept. of Strength of Materials,B-dul Eroilor 29, tel.0268474761, e-mail: rngh@unitbv.ro

Abstract: *The present paper deals with the state of stress and deformation at thin circular plates subjected to symmetrical axial loading. The loading is a bending owing to the uniform distributed loads which act perpendicular to the mean surface of the plate, simultaneous with a membrane load (loads acting in the mean plane of the plate). This type of problem is solved by help of equations which result from the equilibrium of a plate's element and from the boundary and continuity conditions of the mean surface of the plate.*
One considers the following two cases:
the membrane stresses are small comparatively to the bending stresses; in this case the calculus is precise enough if we take into account only the mean plane extensions which will be superposed over the effects given by the transversal bending stress q;
the membrane stresses are considerable and can not be neglected; in this case, second order calculus is required.
Keywords: *deformation, plate, stress, bending, membrane stress*

We further on analyze the plate in Fig. 4, subjected to the transversal loading q and to the forces in the mean plane, forces denoted by $N_r=p$.

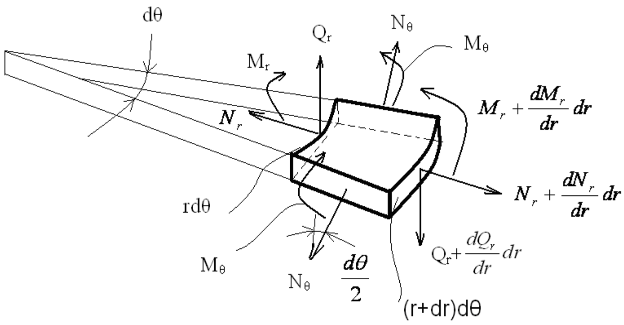

Fig. 7.

The plate's element presented in Fig. 7. is in equilibrium.

$$N_r \cdot rd\theta - (N_r + dN_r)(r+dr)d\theta + 2N_\theta dr \sin\frac{d\theta}{2} = 0$$

$$\left(M_r + \frac{dM_r}{dr}dr\right)(r+dr)d\theta - M_r \cdot rd\theta - 2M_\theta \cdot dr\frac{d\theta}{2} - Q_r \cdot rd\theta \cdot dr = 0 \quad (18)$$

One cancels the analogous terms, we neglect the infinitesimal terms and it successively yields:

$$\frac{N_r - N_\theta}{r} + \frac{dN_r}{dr} = 0 \quad (19)$$

where N_r and N_θ , on basis of Hooke's law and referring to the plate's unit of length, take the following expressions:

$$N_r = \frac{E \cdot h}{1-\nu^2}(\varepsilon_r + \nu \cdot \varepsilon_\theta); \qquad N_\theta = \frac{E \cdot h}{1-\nu^2}(\varepsilon_\theta + \varepsilon_r \cdot \nu) \quad (20)$$

By replacing equations (5) into (20), yields:

$$N_r = \frac{E\cdot h}{1-\nu^2}\left[\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2+\nu\frac{u}{r}\right]; \qquad N_\theta = \frac{E\cdot h}{1-\nu^2}\left[\frac{u}{r}+\nu\frac{du}{dr}+\frac{1}{2}\nu\left(\frac{dw}{dr}\right)^2\right] \tag{21}$$

For the equations of moments one neglects the infinitesimal terms, one divides them to $r{\cdot}d\theta{\cdot}dr$ and the following expression yields:

$$\frac{M_r - M_\theta}{r}+\frac{dM_r}{dr}-Q_r = 0 \tag{22}$$

A sum of projections on vertical direction, taking into account both the curvature of the deformed mean surface of the plate and the loading q leads to a third equation of equilibrium:

$$\frac{d}{dr}(rQ_r)+\frac{d}{dr}\left(rN_r\frac{dw}{dr}\right)+rq = 0 \tag{23}$$

Equations (19), (22) and (23) are equations of equilibrium of the plate, Fig. 4. N_r and N_θ are radial loading and circumference loading, respectively, Q_r is the shear force, M_r and M_θ being the bending moments upon the radius and the circumference.

If one attaches a diametric strip of the plate, subjected to bending, and one considers the loading towards a single direction then, the equation of equivalence between stresses and external loading is (using the notation in Fig. 8):

$$M_i = \int_{-h/2}^{h/2}\sigma_x\cdot z\cdot dz = -\int_{-h/2}^{h/2}\frac{Ez^2}{1-\nu^2}\cdot\frac{d^2w}{dx^2}dz = -D\frac{d^2w}{dx^2} \tag{24}$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ represents the cylindrical stiffness modulus at bending.

Fig. 8.

The well-known equations have been used too:

$$\varepsilon_x = -z\frac{d^2w}{dx^2};\ \varepsilon_x = \frac{\sigma_x}{E}-\nu\frac{\sigma_y}{E};\ \varepsilon_y = \frac{\sigma_y}{E}-\nu\frac{\sigma_x}{E} = 0 \ \text{ yielding: } \ \sigma_x = \frac{E\varepsilon_x}{1-\nu^2}$$

On basis of the deduction method of relation (24) and of the plate's element shown in Fig. 8, the moment's expressions yield:

$$M_x = D\left(\frac{1}{r_x}+\nu\frac{1}{r_y}\right) = -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right); \quad M_y = D\left(\frac{1}{r_y}+\nu\frac{1}{r_x}\right) = -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right) \tag{25}$$

Depending on the curvatures γ_r and γ_θ of the deformed mean surface of the plate, the relations of the moments M_r and M_θ have the expressions below, analogous to Equations (25):

$$M_r = D(\gamma_r+\nu\gamma_\theta); \quad M_\theta = D(\gamma_\theta+\nu\gamma_r) \tag{26}$$

By replacing equations (6) in relations (26) and then in relation (22), the calculus relation of the stresses yields:

$$M_r = -D\left(\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right); \ M_\theta = -D\left(\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2w}{dr^2}\right); \ Q_r = -D\left(\frac{d^3w}{dr^3}+\frac{1}{r}\frac{d^2w}{dr^2}-\frac{1}{r^2}\frac{dw}{dr}\right) \tag{27}$$

The analyzed plates in the present paper respect the condition $\frac{h}{a}<\frac{1}{25}$, being thin plates; in this case one neglects the shear deformations [1].

By integration of the equation of equilibrium (23), we get:

$$Q_r + N_r\frac{dw}{dr}+\frac{1}{2}qr = 0 \tag{28}$$

On basis of relation (5), one establishes a compatibility relation. One follows the succession below:

$$\varepsilon_\theta - \varepsilon_r = \frac{u}{r} - \frac{du}{dr} - \frac{1}{2}\left(\frac{dw}{dr}\right)^2; \quad \frac{d}{dr}(\varepsilon_\theta) = \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{1}{r}\left(\frac{du}{dr} - \frac{u}{r}\right) \text{sau} \ \frac{du}{dr} - \frac{u}{r} = r\frac{d\varepsilon_\theta}{dr};$$

Finally, the relation of compatibility yields:

$$r\frac{d\varepsilon_\theta}{dr} + \varepsilon_\theta - \varepsilon_r + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = 0 \tag{29}$$

By replacing the third relation (27) in relation (28), yield:

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} - \frac{1}{D}N_r\frac{dw}{dr} = \frac{q}{2D}r \tag{30}$$

Relations (20) express the strains as function of forces, yielding the following expressions:

$$\varepsilon_r = \frac{1}{Eh}\left(N_r - \nu N_\theta\right) \quad \text{and} \quad \varepsilon_\theta = \frac{1}{Eh}\left(N_\theta - \nu N_r\right) \tag{31}$$

Based on relations (31), the compatibility relation yields:

$$\frac{dN_\theta}{dr} + \frac{dN_r}{dr} + \frac{Eh}{2r}\left(\frac{dw}{dr}\right)^2 = 0 \tag{32}$$

One replaces the equation of equilibrium (19) in relation (32), yielding:

$$\frac{dN_r}{dr} = -\frac{N_r - N_\theta}{r}; \ \frac{dN_\theta}{dr} - \frac{N_r - N_\theta}{r} + \frac{Eh}{2r}\left(\frac{dw}{dr}\right)^2 = 0 \tag{33}$$

Equations (19), (30) and (33) are three nonlinear equations with three unknowns: dw/dr, N_r and N_θ. The nonlinearity appears in the term N_r, dw/dr (in Eq. 30) and $(dw/dr)^2$ in Eq. 33.

For the problem in Fig. 4, the circular plate is first subjected to stretching by a load of plane stress N_o, around its circumference and then it is subjected to bending under a constant load q. One gets the solution concerning the initial plane stress from the general equations (19)-(27), putting the condition $w=0$ and $q=0$. One yields:

$$N_r = N_\theta = N_0 \qquad \text{and} \qquad u = \frac{N_0}{Eh}(1-\nu)r \tag{34}$$

N_o results from the constant load p.

Equations (34) fulfill the boundary condition $N_r=N_o$, at $r=a$. If $N_r = \sigma_r h$ is the radial load, the number $N_0/Eh = \sigma_r/E = \varepsilon_0$ can be interpreted as the uniaxial load. The plate is first subjected to a stretching under the load N_o and then it is subjected to a vertical load q. The solution yields for each case partly, using the boundary conditions of the plate (conditions for continuity and for relation of the deformed mean surface). For instance, the initial radial load N_o will be affected, $\overline{N}_r$ and $\overline{N}_\theta$ due to the action of the load q.

Depending on the current radius r, in the mean surface the following loads yield:

$$N_r = N_o + \overline{N}_r; \quad N_\theta = N_o + \overline{N}_\theta$$

Equations (19), (30) and (33) are changing accordingly. In the new relations, one may use the following notations:

$$\zeta = \frac{r}{a}; \quad W = \frac{w}{h}; \quad U = \frac{u}{h}; \quad \theta = \frac{dW}{d\zeta} = \frac{a}{h}\cdot\frac{dW}{dr} \tag{35}$$

yielding dimensionless quantities. These are more accessible in engineering. We have to emphasize the fact that such an approach is intricate enough. In general, the analytical solutions are very difficult.

In the case of Fig. 4, the expressions of the stresses will have compulsory components due to the initial, membrane and bending loading, respectively.

The equations presented in the present paper are enough for a complex approach of the circular thin plates bending, in the case of the initial loading in the mean surface. One can also take into account the temperature’s effects, the calculus is reported on the linear theory (small deformations), but particularly on the nonlinear theory (large deformations). The solutions are presented in other papers.

The complexity of the approach of different loading consists in the fact that the solution can not be given dimensionless for each case; that is because of the different dependence of the membrane stress fields on the dimensional parameters. The FEM and FDM present important advantages in the approach of the above problems.

We present further on an example solved by help of FEM (specialized program).

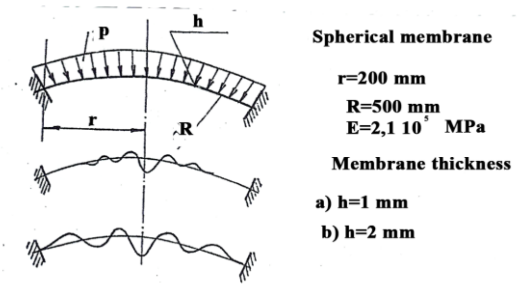

Fig. 9

Fig. 9. presents a fixed spherical membrane subjected to a constant pressure *q*. One divide in 16 finite elements, using the theory presented in the first part of the paper. We have obtained a critical pressure of 1.047 MPa, for h = 1 mm and 4.359 MPa for h=2 mm, respectively. The yielded values are very precise. They have been compared to the values yielded from the equation given by the differential equation made up by help of Bessel's functions (Ponomariov). The equation is:

$$f_w = \frac{Eh^2}{R^2} \cdot \frac{2}{\sqrt{3\left(1-\nu^2\right)}}$$

On basis of the above equation the critical values of the pressure yield: 1.017 MPa and 4.070 MPa, respectively. The eigen ways of loosing stability are shown in the figure above (due to the forking).

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