The 3rd International Conference on "Computational Mechanics and Virtual Engineering"
COMEC 2009
29 – 30 OCTOBER 2009, Brasov, Romania

KINEMATICAL ASPECTS OF THE TRANSMISSION OF MOTION BY CABLE

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Abstract : *The kinematical and dynamical analysis of mechanisms containing transmission by cable necessitates knowing the kinematical equations of the transmission. In most cases the cable is moving in the tangent direction at a fix point of the drum it is winding around. A different case of the transmission of motion by cable is kinematically analysed in this paper by classical methods. This approach implies complicated calculus such as the differentiation with respect to time of complicated functions or performing a lot of algebraic manipulations. These aspects emphasises that would be very helpful to find another method in order to obtain the kinematical equations in a simpler manner. The paper presents the kinematical study by using a hypothetical replacement of the cable with a rigid bar performing a plane motion. The plane motion similarity permits the exploitation of all the properties of this motion and this leads to important simplifications of the calculus.*
Keywords : *cables, kinematical equations, plane motion*

1. INTRODUCTION

There are many devices in engineering based on transmission of motion by cables such as cranes, pulleys, winches, pulley tackles and so on. In order to study kinematics and dynamics of such devices, the kinematical equations of the cable connections must be written. This problem is very simple to solve when the cable is moving in the fix tangent direction of the machine part it is winding around as it can be seen in [1], [2], [3], [4] and [5]. For example one can have two drums connected by a cable as it is shown in figure 1.

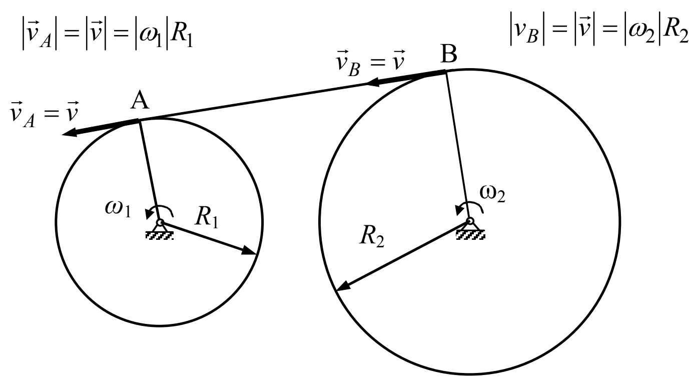

Figure 1: Transmission by cable between two drums.

If the radius of the first drum is denoted R_1 and its angular velocity is denoted ω_1 and the radius of the second drum is denoted R_2 and its angular velocity is denoted ω_2, the kinematical relation of the transmission of motion is

$$|\omega_1| R_1 = |\omega_2| R_2 \quad \textbf{(1)}$$

When the cable is not moving in the tangent direction at a fix point of the drum, the kinematical equations of the transmission of motion are more difficult to obtain. Such a mechanism illustrating this situation is kinematically analysed by classical methods in the next paragraph.

2. OBTAINING THE KINEMATICAL EQUATION BY CLASSICAL METHOD

The mechanism is represented in figure 2. It consists of a drum connected by a cable of a collar that can slide on a fix rod. The radius R of the drum, the distance d and the constant angular velocity ω_0 of the drum are known. The kinematical equation of motion is a relation between the velocity of the collar and the angular velocity of the drum. When the drum rotates, the cable remains tangent at A but the point A is changing its position. In order to better understand this aspect in figure 2 are represented the initial position (when the length of the cable is $A_0B_0=l_0$ and the angle α is ψ) and an intermediate position (when the length of the cable is $AB=l$) of the

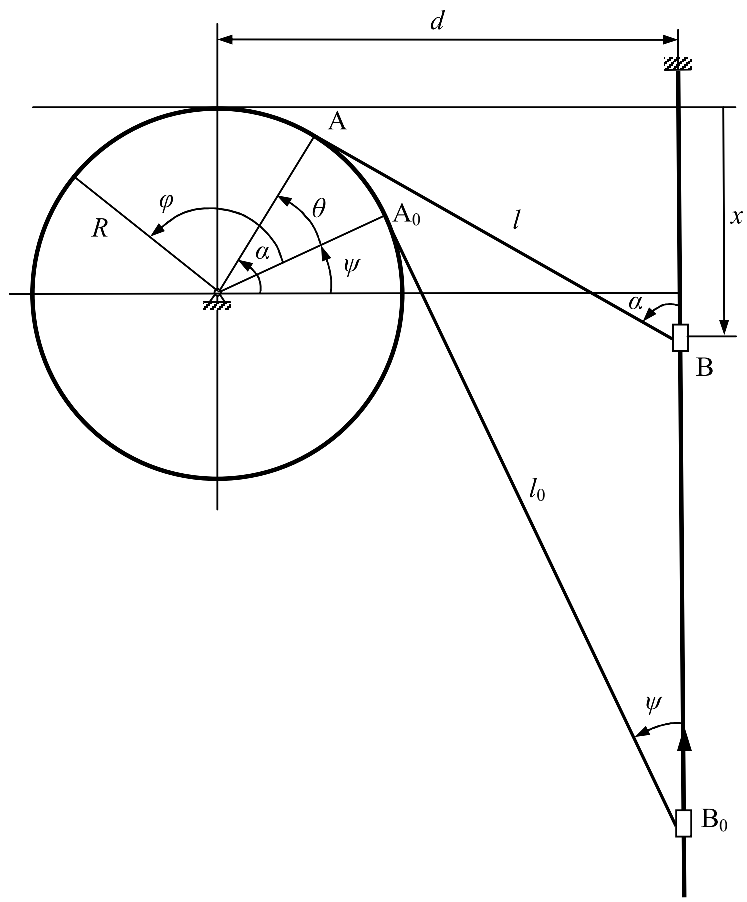

Figure 2: The mechanism at the initial position and at an intermediate position.

mechanism.
When the drum rotates with an angle φ, the cable shortens from the initial length l_0 to the length l and it results the following relation from Fig. 2

$$l_0 = l + R(\varphi - \theta) \tag{2}$$

Knowing that $\dot{\varphi} = \omega_0$, the derivative with respect to time of the equation (2) gives

$$R\omega_0 = R\dot{\theta} - \dot{l} \tag{3}$$

In order to obtain the expressions of $\dot{\theta}$ and $\dot{l}$ as functions of x and $\dot{x}$ the following relations based on figure 2 must be written

$$d = R\cos\alpha + l\sin\alpha \tag{4}$$

$$x - R = -R\sin\alpha + l\cos\alpha \tag{5}$$

By squaring the equations (4) and (5) and adding the results, the following relation is obtained

$$R^2 + l^2 = d^2 + (x - R)^2 \tag{6}$$

The length of the cable is always positive and it results

$$l=\sqrt{d^2-R^2+(x-R)^2} \tag{7}$$

The derivative of the variable length l of the cable with respect to time is

$$\dot{l}=\frac{x-R}{\sqrt{d^2-R^2+(x-R)^2}}\dot{x} \tag{8}$$

In order to obtain $\dot{\theta}$ as a function of x and $\dot{x}$ the equation (4) is solved with respect to the angle α and it results

$$\alpha=\operatorname{asin}\frac{d}{\sqrt{R^2+l^2}}-\operatorname{atan}\frac{R}{l} \tag{9}$$

Based on figure 2 we can write

$$\theta=\alpha-\psi \tag{10}$$

Because the angle ψ is the initial value of the angle α, the derivative with respect to time of the equation (10) yields

$$\dot{\theta}=\dot{\alpha} \tag{11}$$

By performing the derivative of the equation (9) with respect to time, we obtain

$$\dot{\theta}=\left(\frac{R}{R^2+l^2}-\frac{ld}{\left(R^2+l^2\right)\sqrt{R^2+l^2-d^2}}\right)\dot{l} \tag{12}$$

After replacing the equation (12) in the equation (3) it results

$$R\omega_0=-\frac{l^2\sqrt{R^2+l^2-d^2}+Rld}{\left(R^2+l^2\right)\sqrt{R^2+l^2-d^2}}\dot{l} \tag{13}$$

By using the equations (7) and (8) and after some algebraic manipulations from the equation (13) the following final form of the kinematical equation of motion results

$$\dot{x}=-\frac{R[d^2+(x-R)^2]}{(x-R)\sqrt{d^2+(x-R)^2-R^2}+dR}\omega_0 \tag{14}$$

The expression given by the equation (14) is not very simple as the equation (1) is, and many algebraic manipulations are necessary in order to obtain it. For this reason it is needed to find out a simpler method to obtain this result.

3. THE SIMILARITY BETWEEN THE MOTION OF A CABLE AND THE PLANE MOTION OF A RIGID BAR

A similar mechanism where the cable is replaced by a rigid bar and the drum with a circular disc is represented in figure 3. There is no slipping between the disc and the bar. We can consider the disc as being a toothed gear and the bar as being a rack bar for example. The bar and the collar are connected by a hinge. When the disc rotates with constant angular velocity ω_0, the bar performs a plane motion that we suppose it is kinematically similar to the motion of the cable. The instantaneous centre of rotation of the bar, denoted by I, is located at the intersection of the perpendiculars to the velocity of the point A and of the point B of the bar as it can be seen in figure3. The direction of the angular velocity ω of the bar must comply with the direction of the velocity of the points A and B (see figure 3). Because the point A is on the rim of the circle of radius R, its velocity is

$$|\vec{v}_A|=|\vec{\omega}_0|R \tag{15}$$

The velocity of the point A as a point of the bar in plane motion is

$$|\vec{v}_A|=|\vec{\omega}||IA| \tag{16}$$

The velocity of the point B as a point of the bar in plane motion is

$$|\vec{v}_B|=|\vec{\omega}||IB| \tag{17}$$

After replacing the equations (16) and (17) in the equation (15) results

$$|\vec{v}_B|=|\vec{\omega}_0|R\frac{|IB|}{|IA|} \tag{18}$$

In the right angle triangle ABI take place the relation $|IA|=|IB|\cos\alpha$ and equation (18) becomes

$$|\vec{v}_B| = \frac{|\vec{\omega}_0| R}{\cos\alpha} \quad \textbf{(19)}$$

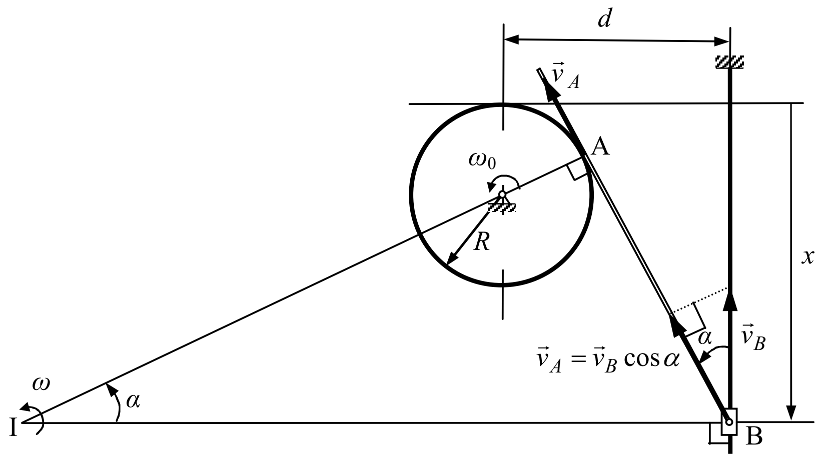

Figure 3: The mechanism with the cable replaced by a rigid bar

The same result can be immediately obtained based on the well known theorem: *the projections of the velocities of the points of a line segment in plane motion on that segment are equal.* By applying this theorem to the points *A* and *B* of the bar *AB* from figure 3 it results

$$|\vec{v}_A| = |\vec{v}_B| \cos\alpha \quad \textbf{(20)}$$

The equation (19) results immediately by taking into account the equation (15).
In order to get the cosine of the angle α as a function of x we rewrite equation (4) under the form

$$d - R\cos\alpha = l\sqrt{1-\cos^2\alpha} \quad \textbf{(21)}$$

The solution is

$$\cos\alpha = \frac{dR + l\sqrt{l^2 + R^2 - d^2}}{R^2 + l^2} \quad \textbf{(22)}$$

After replacing the equation (7) in the equation (22) the cosine of the angle α becomes

$$\cos\alpha = \frac{(x-R)\sqrt{d^2 + (x-r)^2 - R^2} + Rd}{d^2 + (x-R)^2} \quad \textbf{(23)}$$

By replacing the equation (23) in the equation (19) the absolute value of the velocity of the point *B* is obtained

$$|\vec{v}_B| = \frac{R[d^2 + (x-R)^2]}{(x-R)\sqrt{d^2 + (x-R)^2 - R^2} + dR}\,\omega_0 \quad \textbf{(24)}$$

The absolute value of the velocity of the point *B* obtained by considering the plane motion of a rigid bar instead of the cable is the same as that obtained by using the classical calculus. This proves the perfect similarity between the motion of the cable and the plane motion of a rigid bar. It is more convenient to use the model of plane motion of a bar because there are many properties of the plane motion that can be used to simplify the calculus. For example there is no need to perform the derivatives of some complicated expression with respect to time and the algebraic manipulations are simpler. The method can be applied to other mechanisms of the same kind.

3. EXAMPLE OF USING THE SIMILARITY TO ANOTHER MECHANISM

Another kind of mechanism containing a cable is depicted in figure 4. It consists of a drum and a bar connected by a cable. The drum of radius *R* rotates with constant angular velocity ω_0. The bar of length *L* rotates with the angular velocity ω about the end O where a fix hinge is. A cable is winding around the drum and its end is connected to the other end B of the bar. The length AB of the cable at an intermediate moment is denoted by *l*. It must be found the angular velocity ω as a function of the position parameter φ that is the angle between the bar

OB and the vertical direction (see figure 4). In order to do this, we hypothetically replace the cable by a rigid bar performing a plane motion. The velocity of the point A is

$$|\vec{v}_A| = |\vec{\omega}_0| R \quad \textbf{(25)}$$

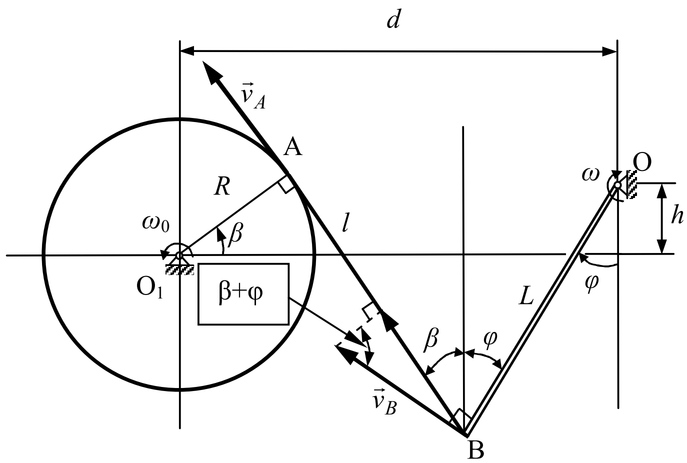

Figure 4: A mechanism of the same kind containing a cable

The velocity of the point B is

$$|\vec{v}_B| = |\vec{\omega}| L \quad \textbf{(26)}$$

In accordance with the above theorem, the projection of the velocity of the point B on the direction AB equals the velocity of the point A, that is

$$|\vec{v}_A| = |\vec{\omega}_0| R = |\vec{v}_B| \sin(\beta + \varphi) = |\vec{\omega}| L \sin(\beta + \varphi) \quad \textbf{(27)}$$

where β is the angle of the radius O_1A with the horizontal direction.
From the equations (27) it results

$$|\vec{\omega}| = \frac{|\vec{\omega}_0| R}{L \sin(\beta + \varphi)} \quad \textbf{(28)}$$

The following relations result from the figure 4

$$R\cos\beta + l\sin\beta + L\sin\varphi = d \quad \textbf{(29)}$$

$$R\sin\beta - l\cos\beta + L\cos\varphi = h \quad \textbf{(30)}$$

The equations (29) and (30) can be written

$$R\cos\beta + l\sin\beta = d - L\sin\varphi \quad \textbf{(31)}$$

$$R\sin\beta - l\cos\beta = h - L\cos\varphi \quad \textbf{(32)}$$

By squaring the equations (31) and (32) and adding the results the following formula for the length l of the bar (cable) is obtained

$$l = \sqrt{L^2 + h^2 + d^2 - R^2 - 2ld\sin\varphi + 2Lh\cos\varphi} \quad \textbf{(33)}$$

Now we multiply first the equation (31) by sinφ and the equation (32) by cosφ and then we add the results in order to obtain the following equation

$$R\sin(\beta + \varphi) - l\cos(\beta + \varphi) = h\cos\varphi + d\sin\varphi - L \quad \textbf{(34)}$$

which can be written

$$R\sin(\beta + \varphi) - l\sqrt{1 - (\sin(\beta + \varphi))^2} = h\cos\varphi + d\sin\varphi - L \quad \textbf{(35)}$$

Taking into account the relation (33), the convenient solution of the previous equation having the unknown sin(β+φ) is

$$\sin(\beta + \varphi) = \frac{R(h\cos\varphi + d\sin\varphi - L) + |d\cos\varphi - h\sin\varphi| \sqrt{L^2 + h^2 + d^2 - R^2 - 2ld\sin\varphi - 2Lh\cos\varphi}}{L^2 + h^2 + d^2 - 2ld\sin\varphi - 2Lh\cos\varphi} \quad \textbf{(36)}$$

The kinematical equation (28) becomes

$$|\vec{\omega}| = \frac{|\vec{\omega}_0| R(L^2 + h^2 + d^2 - 2ld\sin\varphi - 2Lh\cos\varphi)}{L\left[R(h\cos\varphi + d\sin\varphi - L) + |d\cos\varphi - h\sin\varphi| \sqrt{L^2 + h^2 + d^2 - R^2 - 2ld\sin\varphi - 2Lh\cos\varphi} \right]} \quad \textbf{(37)}$$

Even this relation is complicated the algebraic manipulations necessary to deduce it are not so complex and the similarity with the plain motion of a bar offer the easier way to obtain it.

3. CONCLUSIONS

The motion transmission by cables is very much used in mechanical engineering and the kinematical equations of the cable connections must be written in order to study kinematics and dynamics of the devices containing such kind of transmissions of motion. The kinematical equation of the cable connection is easy to be determined when the cable is moving in a fix tangent direction of the drum it is winding around. In this situation the mathematical model of the connection is a very simple expression and very easy to deduce. Some difficulties can occur when the cable is not moving in the tangent direction at a fix point of the drum. The kinematical equations are more difficult to obtain in such cases. The paper presents first the classical method to deduce the kinematical equation of such a cable connection emphasising the complexity of the mathematical calculus and the necessity of finding a simpler method in order to deduce it. Then the paper proves the perfect similarity between the motion of a cable and the plane motion of a rigid bar. The approach based on the plane motion permits to use all the properties of the plane motion of a rigid body for studying the motion of the cable.
Some important simplifications of the calculus, such as not performing the derivatives of some complicated expression with respect to time or important simplifications of algebraic manipulations, which were otherwise necessary in the classical approach, result by using the model of the plane motion of a bar.
This similarity can be very helpful in order to study many other mechanisms of the same kind in order to obtain the kinematical relations necessary to study the kinematics and dynamics of those mechanisms. The paper presents an example of such a mechanism containing a cable. Based on the similarity with the plane motion of a rigid bar the kinematical equation is deduced.

REFERENCES

[1] Meriam J. L., Kraige L. G.: Engineering Mechanics: Dynamic, 6th edition, John Wiley & Sons, New York, 2007.
[2] Beer F. P., Russel E. J., Clausen E. W.: Vector Mechanics for Engineers. Dynamics, 8th Edition, Mc. Graw-Hill, Singapore, 2007.
[3] Hibbeler R. C., Engineering Mechanics: Dynamic, 11th Edition, Prentice Hall, Bangkok, 2007.
[4] Millard F. B. Jr.: Principles of Engineering Mechanics, Volume 1, Kinematics – The Geometry of Motion, Plenum Press, New York and London, 1986.
[5] Millard F. B. Jr.: Principles of Engineering Mechanics, Volume 2, Dynamics – The Analysis of Motion, Plenum Press, New York and London, 2006.