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ANALYTICAL METHODS FOR THE SOLUTION OF THE PLANE PROBLEM OF THE THERMOELASTIC EQUILIBRIUM

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Abstract: *In this paper, analytical methods for the solution of the plane problem of the thermoelastic equilibrium (PPTE) by means of complex functions are presented. Supposing that the thermal complex potential* $t(z,\bar{z})$ *and the thermoelastic displacement potential* $\varphi(z,\bar{z})$ *are determined, one can compute the displacements, the deformations and the stress state in plates (plane sections)* $(u_i, \varepsilon_{ij}, T_{ij})$*. Two problems are presented: the decoupled problem* (P1) *when the states* $(u_i, \varepsilon_{ij}, T_{ij})$ *are generated by the thermal field* T *on the boundary (i.e. without mechanical loadings) and the coupled problem* (P2) *when the medium is subjected to thermal stress and mechanical loadings on the boundary. These studies made for the canonical domains (half-plane or circle) lead to boundary value problems with special boundary conditions for harmonic or biharmonic functions. These methods can also be applied to composite media. At the end of this paper, applications to half-plane or circle are presented. For more general domains, conformal transforms can be applied.*
Keywords: *thermoelastic, thermal potential, potential of displacements, biharmonic problem, analytical solutions.*

1. THE PLANE PROBLEM OF THE THERMOELASTIC EQUILIBRIUM (PPTE)

Consider a homogeneous isotropic elastic medium in a state of plane thermoelastic deformation which parallel to the x_1Ox_2 plane, i.e. a state in which the displacement $\vec{u}$, the deformation (ε_{ij}), the stress (T_{ij}) and the temperature T depend only on (x_1, x_2) in the stationary case, on every cross section $x_3 = const$

$$u_i = u_i(x_1, x_2),\ u_3 = 0,\ \varepsilon_{ij} = \varepsilon_{ij}(x_1, x_2),$$
$$T_{ij} = T_{ij}(x_1, x_2),\ T = T(x_1, x_2),\ i, j = 1,2,\ (x_1, x_2) \in D, \tag{1}$$

where D is a bounded domain in x_1Ox_2 with the boundary denoted by C. The constitutive equations, the equations of thermoelastic equilibrium and the compatibility conditions are (see [2], [3], [5], [6])

$$T_{ii} = \frac{E}{2(1-\nu^2)}(\varepsilon_{ii} - \nu\varepsilon_{jj}) - \frac{E\alpha T}{1+\nu},\ T_{12} = \frac{E}{2(1+\nu)}\varepsilon_{12},$$
$$\varepsilon_{ii} = \frac{1}{E}(T_{ii} - \nu T_{jj}) + \alpha T,\ \varepsilon_{12} = \frac{2(1+\nu)}{E}T_{12}, \tag{2}$$
$$2\varepsilon_{ij} = u_{i,j} + u_{j,i},\ \varepsilon_{ii} = u_{i,i},\ i, j = 1,2,$$

$$\mu u_{i,jj} + (\lambda + \mu)u_{j,ij} - \alpha T_{,i} = 0,\ \varepsilon_{11,22} + \varepsilon_{22,11} = 2\varepsilon_{12,12},$$
$$\mu\Delta u_i + (\lambda + \mu)\theta_{,i} - \alpha T_{,i} = 0,\ \theta = u_{i,i} = \varepsilon_{11} + \varepsilon_{22}, \tag{3}$$

where E, ν, λ, μ are the elastic coefficients and α is the thermoelastic constant of deformation.

Generally, for the equilibrium equations $T_{ij},_j = 0$ and the temperature $T(x,y)$, the following loading conditions on the boundary are associated

$$T_{ij} n_j \big|_C = \hat{t}_i^*(x,y),\ T(x,y)\big|_C = T^*(x,y),\ i,j = 1,2. \quad \textbf{(4)}$$

Let $U(x,y) \in C^4(D)$ be the Airy stress function; thus

$$T_{ij} = U,_{kk} \delta_{ij} - U,_{ij},\ i,j = 1,2. \quad \textbf{(5)}$$

By using $T_{ij},_j = 0$, (2) and (3), we obtain the equation (see [3], [5], [6])

$$\Delta\Delta U + \alpha E \Delta T = 0 \Leftrightarrow \Delta(\Delta U + \alpha E T) = 0 \quad \textbf{(6)}$$

which gives solutions to problem (P1):

$$\Delta\Delta U = 0,\ U\big|_C = 0,\ \frac{\partial U}{\partial n}\bigg|_C = 0,\ \Delta T = 0,\ T\big|_C = T^*(x,y). \quad \textbf{(7)}$$

For $\alpha = 0$, we have the plane problem of the elasticity (PPE) (i.e. without changing of heat).
Assuming, in the absence of the mass forces, that $\operatorname{rot}\vec{u} = 0$, then there exists the thermoelastic potential of displacements $\Phi(x,y)$, so that

$$\vec{u} = \operatorname{grad}\Phi,\ \operatorname{div}\vec{u} = \theta = \Delta\Phi,\ u_i = \Phi,_i,\ i = 1,2. \quad \textbf{(8)}$$

By substituting (8) in (4), we get (according to [4])

$$(1-\nu)\frac{\partial}{\partial x_i}(\Delta\Phi) = (1+\nu)\alpha T,_i,\ \Delta\Phi = \frac{1+\nu}{1-\nu}\alpha T(x,y) \text{ (Poisson)} \quad \textbf{(9)}$$

If we find from (7) the temperature distribution $T(x,y)$ in D, the Poisson equation (9) has the solution

$$\Phi(x,y) = \frac{1}{2\pi}\frac{1+\nu}{1-\nu}\alpha \iint_D T(\xi,\eta) \ln r\, d\xi\, d\eta \text{, for all } (x,y) \in D, \quad \textbf{(10)}$$

where $r = (\xi - \eta)^2 + (\eta - y)^2$. Taking into account the complex representation $x = \frac{z + \bar{z}}{2}$, $y = \frac{z - \bar{z}}{2i}$, we can construct by using (10) the potentials $T(x,y) = t(z,\bar{z})$ and $\Phi(x,y) = \varphi(z,\bar{z})$. Hence, equation (9) becomes

$$\Delta\Phi = 4\frac{\partial^2\varphi}{\partial z \partial\bar{z}} = \frac{1+\nu}{1-\nu}\alpha t(z,\bar{z}) \quad \textbf{(11)}$$

and its solution (10) becomes

$$\varphi(z,\bar{z}) = \alpha\beta \iint_D t(z,\bar{z}) dz\, d\bar{z}, \quad \textbf{(12)}$$

with $4\beta = \frac{1+\nu}{1-\nu}$. Due to the linearity, we will split problem (P1) into two applications (A') $(u_i', \varepsilon_{ij}', T_{ij}', U')$ and (A'') $(u_i'', \varepsilon_{ij}'', T_{ij}'', U'')$, where

$$u_i = u_i' + u_i'',\ \varepsilon_{ij} = \varepsilon_{ij}' + \varepsilon_{ij}'',\ T_{ij} = T_{ij}' + T_{ij}'',\ \hat{t}_i^* = t_i' + t_i'',\ U = U' + U''. \quad \textbf{(13)}$$

For application (A'), we find $T(x,y) = t(z,\bar{z})$ from (7) and using (12), we compute $\varphi(z,\bar{z}) = \Phi(x,y)$. Now, we can find u_i', ε_{ij}' by using $u_i' = \Phi,_i$ and according to (2), we find

$$T_{ij}' = 2\mu(\Phi,_{ij} - \Phi,_{rr} \delta_{ij}),\ i,j = 1,2.$$

However, following the idea of Goodier-Lebedev [6], by introducing the potentials $t(z,\bar{z})$ and $\varphi(z,\bar{z})$, one can formally verify some relations of Kolosov-Mushelishvilii type (see [2], [3]) and, solving (A'), we have

$$u_1' + u_2' = \alpha\beta \int t(z,\bar{z}) dz,\ T_{11}' + T_{22}' = -4\alpha\beta\mu\, t(z,\bar{z}),$$

$$T_{22}' - T_{11}' + 2iT_{12}' = 4\alpha\beta\mu \int \frac{\partial}{\partial\bar{z}} t(z,\bar{z}) dz. \quad \textbf{(14)}$$

For application (A''), we will transform problem (PPTE) into a problem (PPE) for U'' (see (15) below). By considering with (7) the absence of the stress caused by loadings for problem (P1), loadings for U'' due to U' will appear. In view of $U\big|_C = 0 = U'\big|_C + U''\big|_C$, $\left.\frac{\partial U}{\partial n}\right|_C = 0 = \left.\frac{\partial U'}{\partial n}\right|_C + \left.\frac{\partial U''}{\partial n}\right|_C$, we obtain the problem

$$\Delta\Delta U'' = 0\,,\; U''\big|_C = -U'\big|_C = R_1\,,\; \left.\frac{\partial U''}{\partial n}\right|_C = -\left.\frac{\partial U'}{\partial n}\right|_C = R_2\,, \tag{15}$$

where R_1, R_2 are given by U' on C.

2. THERMAL POTENTIAL

Consider a plane heat field which act in D

$$\vec{Q}(x,y) = Q_x\vec{i} + Q_y\vec{j}\,, \tag{16}$$

where $\vec{Q}$ is a potential which satisfies

$$\operatorname{div}\vec{Q} = 0\,,\; \operatorname{rot}\vec{Q} = 0\,. \tag{17}$$

We have

$$\vec{Q} = \operatorname{grad} T(x,y), \tag{18}$$

where $T(x,y)$ is the thermal potential (temperature) and $T(x,y) = c$ are isothermal curves. It follows from (17) that

$$\Delta T(x,y) = 0 \text{ in } D\,, \tag{19}$$

which is the heat equation. Let $H(x,y)$ be the harmonic conjugate of $T(x,y)$. The relation between T and H is given by the Cauchy-Riemann relations. Thus, $\Delta H(x,y) = 0$ in D, and $H(x,y) = k$ are thermal field lines for which $\vec{Q}$ are tangent vectors. Now, we construct the holomorphic function

$$W(z) = T(x,y) + iH(x,y), \text{ with } T = \Re W(z). \tag{20}$$

By using the complex heat potential W, we have (see [6])

$$u_1' + iu_2' = \alpha\beta\int W(z)dz = \alpha\beta\int t(z,\bar{z})dz \text{ in } D\,. \tag{21}$$

By using (15) with $T_{ij} = T_{ij}'' + 2\mu(\Phi,_{ij} - \Phi,_{rr}\,\delta_{ij})$ and $\hat{t}_i^*\big|_C = 0 = \hat{t}_i'\big|_C + \hat{t}_i''\big|_C$, we obtain the following conditions for U''

$$\hat{t}_i''\big|_C = -2\mu(\Phi,_{ij}\, n_j - \Phi,_{rr}\, n_i)\big|_C\,, \tag{22}$$

i.e.

$$\hat{t}_1''\big|_C = \frac{d}{ds}(U'',_2) = 2\mu\frac{d}{ds}(\Phi,_2) = l_1\,,\; \hat{t}_2''\big|_C = -\frac{d}{ds}(U'',_1) = -2\mu\frac{d}{ds}(\Phi,_1) = l_2\,. \tag{23}$$

Therefore, knowing $\Phi(x,y)$ and using (22) and (23), we can find $U'',_1$ and $U'',_2$; thus $U''\big|_C$ and $\left.\frac{\partial U''}{\partial n}\right|_C$ can be found. Let

$$C: x = x(s),\; y = y(s),\; 0 \le s \le l\,,$$

be the arc-length parameterization of the curve C. Then, $\vec{t}\left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ and $\vec{n}\left(\frac{dy}{ds}, -\frac{dx}{ds}\right)$ are the unit tangent vector and the unit outer normal vector to the curve C, respectively. In this case of (PPE) we have $\Delta\Delta U'' = 0$ in D and the boundary conditions (see [4])

$$U''\big|_C = 2\mu\Phi(s) = R_1(x,y),\; \left.\frac{\partial U''}{\partial n}\right|_C = 2\mu\left.\frac{\partial \Phi}{\partial n}\right|_C = R_2(x,y). \tag{24}$$

By solving the problem (15) with the boundary conditions (24) we find U'' and then we can compute the parameters for application (A''): $T''_{11} = U'',_{22}$, $T''_{22} = U'',_{11}$, $T''_{12} = -2U'',_{12}$ and ε''_{ij}, u''_i can be computed with the help of (2).

Remark. In the case of problem (P2) when the thermal boundary conditions $T^*(x,y)$ are coupled with the given mechanical loadings $\hat{t}_i^*(x,y)$, the argument is the same by adding to $\hat{t}''_i\big|_C$ in (23) the data $\hat{t}_i^*(x,y)$, i.e. $\hat{t}''_i\big|_C = l_i + t_i^*(x,y)$. Consequently, the method is based on the calculus of the potentials $t(z,\bar{z})$, $\varphi(z,\bar{z})$.

3. THE BOUNDARY VALUE PROBLEMS FOR THE CANONICAL DOMAINS HALF-PLANE AND CIRCLE

3.1. The Dirichlet problem in a half-plane (DPH)

We seek for a holomorphic function $f(z) = U(x,y) + iV(x,y)$ in the upper half-plane $D^+ : y > 0$, knowing that

$$\Delta U(x,y) = 0 \text{ in } D^+,$$

$$U(x,y)\big|_{y=0} = \Re f(z)\big|_{y=0} = U^*(x).$$

By Cisotti's formula we have the solution of (DPH)

$$f(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{U^*(t)}{t-z}dt + ik\,,\ z \in D^+.$$

If $U^*(x)$ is a rational function, we form the complex function $U^*(z) = P_p^u U^*(z) + P_p^l U^*(z)$, where $P_p^u U^*(z)$, $P_p^l U^*(z)$ are the principal parts of $U^*(z)$ for the upper half-plane D^+ and for the lower half-plane $D^- : y < 0$, respectively. Then $f^u(z) = 2P_p^l U^*(z)$.

Problem (DPH) with rational data on portions of the boundary is

$$\Re f(z)\big|_{y=0} = \begin{cases} g(x)\ on\ (a,b), \\ 0\ on\ (-\infty,\infty)\setminus(a,b), \end{cases}$$

where $g(x)$ is a rational function. The solution of this problem is

$$f(z) = g(z)G(z) - \left\{P^+(z) - \overline{P^-(\bar{z})}\right\},$$

where $G(z) = \dfrac{1}{\pi i}\ln\dfrac{z-b}{z-a}$ and $P^+(z) = P_p^u\{g(z)G(z)\}$, $P^-(z) = P_p^l\{g(z)G(z)\}$.

3.2. The fundamental biharmonic problem in a half-plane (FBPH)

Consider the boundary value problem [4]

$$\Delta\Delta U = 0 \text{ in } D^+,$$

$$U\big|_C = R_1(x),\ \frac{\partial U}{\partial n}\bigg|_C = -\frac{\partial U}{\partial y}\bigg|_C = R_2(x),$$

where $C : y = 0$ is the boundary of D^+. We seek the solution of (FBPH) under the form $U(x,y) = \Re F(z)$, where $F(z) = A(z) + yB(z)$ and the functions $A(z)$, $B(z)$ are holomorphic in D^+. The solution to (FBPH) will be

$$A(z) = \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{R_1(t) - R_2(t)}{t-z}dt + ik_1,\ B(z) = -iA'(z) - \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{R_2(t)}{t-z}dt + ik_2,\ z \in D^+.$$

Problem (FBPH) with rational data on portions of the boundary can be discussed as in Subsection 3.1 (see [4]).

3.3. The Dirichlet problem for a circle (DPC)

We seek for a holomorphic function $f(z)=U(x,y)+iV(x,y)$ in the disk $D^i:|z|<a$, knowing that

$$\Delta U(x,y)=0 \text{ in } D^i,$$

$$U(x,y)\big|_C=U^*(x,y)=U^*\left(\frac{1}{2}\left(z+\frac{a^2}{z}\right),\frac{1}{2i}\left(z-\frac{a^2}{z}\right)\right),$$

where $C:|z|=a$ is the boundary of D^i. The solution of (DPC) is given by the Schwarz-Villat formula

$$f(z)=\frac{1}{2\pi i}\oint_C\frac{U^*(\zeta)(\zeta+z)}{\zeta(\zeta-z)}d\zeta+ik,\ z\in D^i.$$

If $U^*(z)$ is a rational function, we form the complex function $U^*(z)=P_p^iU^*(z)+P_p^eU^*(z)$, where $P_p^iU^*(z)$, $P_p^eU^*(z)$ are the principal parts of $U^*(z)$ for the interior of the circle D^i and for the exterior of the circle $D^e:|z|>a$, respectively. Then $f^i(z)=2P_p^eU^*(z)$.

Problem (DPC) with rational data on portions of the boundary is (see [1])

$$U\big|_C=\begin{cases}g_0(x,y)\ on\ C_h,\\ 0\ on\ C\setminus C_h,\end{cases}$$

where $g_0(x,y)$ is a rational function and $C_h=\{z=ae^{i\theta}:\theta_h<\theta<\theta_{h+1}\}$ is a portion of C. The solution of this problem is

$$f(z)=g_h(z)G(z)-\left\{P(z)-\overline{P\left(\frac{a^2}{\bar z}\right)}\right\},$$

where $G(z)=-\dfrac{1}{2\pi}(\theta_{h+1}-\theta_h)+\dfrac{1}{\pi i}\ln\dfrac{ae^{i\theta_{h+1}}-z}{ae^{i\theta_h}-z}$, $g_h(z)=g_0(x,y)G(z)$, $P(z)=P_p\{g_0(x,y)G(z)\}$.

3.4. The fundamental biharmonic problem for a circle (FBPC)

Consider the boundary value problem [1]

$$\Delta\Delta U=0 \text{ in } D^i,$$

$$U\big|_C=R_1(x,y)=R_1\left(\frac{1}{2}\left(z+\frac{a^2}{z}\right),\frac{1}{2i}\left(z-\frac{a^2}{z}\right)\right),$$

$$\frac{\partial U}{\partial n}\bigg|_C=R_2(x,y)=R_2\left(\frac{1}{2}\left(z+\frac{a^2}{z}\right),\frac{1}{2i}\left(z-\frac{a^2}{z}\right)\right),$$

where $C:|z|=a$ is the boundary of $D^i:|z|<a$. We seek the solution of (FBPC) under the form $U(x,y)=\Re F(z)$, where $F(z)=A(z)+(a^2-z\bar z)B(z)$ and the functions $A(z)$, $B(z)$ are holomorphic in D^i. The solution to (FBPC) will be

$$A(z)=\frac{1}{2\pi}\oint_C\frac{R_1(\zeta)(\zeta+z)}{\zeta(\zeta-z)}d\zeta+ik_1,\ B(z)=\frac{z}{2a^2}A'(z)+\frac{1}{2\pi i}\oint_C\frac{R_2(z)(\zeta+z)}{\zeta(\zeta-z)}d\zeta+ik_2,\ z\in D^i.$$

4. APPLICATIONS

Application 1. Consider problem (P1) for the half-plane $D^+:y>0$ where the boundary $C:y=0$ is subjected to thermal stress and there are no mechanical loadings. More precisely, we have the following boundary conditions

$$T(x,y)\big|_{y=0} = \begin{cases} T_0 = const, \; if \;\; x \in (-1,1), \\ 0, \; if \;\; x \in (-\infty,\infty) \setminus (-1,1). \end{cases}$$

According to Subsection 3.1 and (12), we have

$$W(z) = T(x,y) + iH(x,y) = \frac{T_0}{\pi i} \ln \frac{z-1}{z+1},$$

$$T(x,y) = \Re W(z) = \frac{T_0}{\pi} \operatorname{arctg} \frac{2y}{x^2+y^2-1}, \; t(z,\bar{z}) = -\frac{T_0}{\pi} \operatorname{arctg} \frac{z-\bar{z}}{z\bar{z}-1},$$

$$\varphi(z,\bar{z}) = \frac{T_0 \alpha \beta}{\pi} \left\{ \frac{i\bar{z}}{2} \ln\left(z^2-1\right) - \frac{iz}{2} - z\bar{z} \operatorname{arctg} \frac{i(z-\bar{z})}{z\bar{z}-1} \right\},$$

$$\Phi(x,y) = \frac{T_0 \alpha \beta}{\pi} \left\{ \left(\mathrm{x}^2 + y^2\right) \operatorname{arctg} \frac{2y}{x^2+y^2-1} - x \operatorname{arctg} \frac{2xy}{x^2-y^2-1} \right.$$
$$\left. + \, y \ln \sqrt{\left(x^2-y^2-1\right)^2 + 4x^2y^2} \right\}.$$

Then the biharmonic problem can be solved.

Application 2. Consider the thermoelastic problem (P1) in the interior $D^i : |z| < 1$ of the unit circle. The boundary conditions for the harmonic function $T(x,y)$ in D^i are

$$T(x,y)\big|_{C} = \begin{cases} T_0 = const, \; if \;\; \theta \in (0,\pi), \\ 0, \; if \;\; \theta \in (\pi, 2\pi), \end{cases}$$

where $C = \partial D^i = \left\{ z = e^{i\theta} : \theta \in [0,2\pi] \right\}$. According to Subsection 3.3 we have

$$W(z) = T(x,y) + iH(x,y) = -\frac{T_0}{2} - \frac{iT_0}{\pi} \ln \frac{z+1}{z-1},$$

$$T(x,y) = \Re W(z) = -\frac{T_0}{2} - \frac{T_0}{\pi} \operatorname{arctg} \frac{2y}{x^2+y^2-1}, \; t(z,\bar{z}) = -\frac{T_0}{2} + \frac{T_0}{\pi} \operatorname{arctg} \frac{i(z-\bar{z})}{z\bar{z}-1},$$

$$\varphi(z,\bar{z}) = T_0 \alpha \beta \left\{ \frac{1}{\pi} \operatorname{arctg} \frac{i(z-\bar{z})}{z\bar{z}-1} - \frac{z\bar{z}}{2} + \frac{iz}{2\pi} \ln\left(\bar{z}^2-1\right) - \frac{i\bar{z}}{2\pi} \ln\left(z^2-1\right) \right\},$$

$$\Phi(x,y) = T_0 \alpha \beta \left\{ \frac{1}{\pi} \left[x \operatorname{arctg} \frac{2xy}{x^2-y^2-1} - y \ln \sqrt{\left(x^2-y^2-1\right)^2 + 4x^2y^2} \right] \right.$$
$$\left. - \frac{1}{2}\left(\mathrm{x}^2+y^2\right) - \frac{1}{\pi}\left(\mathrm{x}^2+y^2\right) \operatorname{arctg} \frac{2y}{x^2+y^2-1} \right\}.$$

Then the biharmonic problem can be solved.

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