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A GENERAL METHOD TO STUDY THE MOTION IN A NON-INERTIAL REFERENCE FRAME

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Abstract: *The paper offers a general method to study the motion of a mass particle with respect to a non-inertial reference frame. By using an adequate tensor instrument, we obtain a simplified form of the initial value problem that models the non-inertial motion. The study of the motion in a non-inertial reference frame in a central positional force field is then reduced to the study of the classic motion in a central force field. The applications to this method are in solving Kepler's problem in non-inertial reference frames, solving the relative orbital motion problem and deriving the explicit solution to a classic Theoretical Mechanics problem: The Foucault Pendulum.*
The advantage of using this method in the study of the motion in a non-inertial reference frames is that is significantly reduces the amount of computations and it offers, in some important particular cases, closed form analytical solutions.
Keywords: *Non-inertial frame, Orthogonal tensors, Kepler's problem, Foucault Pendulum, Central force field.*

1. INTRODUCTION

The motion in a non-inertial reference frame has important applications, both in theoretical and applied scientific problems. The present approach offers a general method to study such types of motion, with successful applications in some relevant classical and modern problems: the Kepler problem in a rotating reference frame, the Foucault Pendulum problem, the relative orbital motion problem.
The tensor instrument that simplifies the complex initial value problem that models the motion was introduced for the first time in 1995 by D. Condurache [1] and it was used to approach several problems of Classical and Celestial Mechanics. The tensor operator which is presented in this paper is introduced by the Darboux equation [2], written here in its tensorial form [3]. In most situations discussed here, this operator has a time-explicit formula and it allows determining explicit or closed form vectorial expressions for the relative law of motion and the relative velocity.
The motion in a central positional force field with respect to a rotating frame is briefly discussed, and an algorithm for a potential approach is proposed. The study of such type of motion leads to the three applications which are presented in the paper: the Kepler problem in rotating reference frames, the Foucault Pendulum and the relative orbital motion in a gravitational force field. The equations of the relative motion of satellites are presented here in a vectorial coordinate-free form.

2. PROBLEM FORMULATION

The motion of a particle with respect to a rotating reference frame is described by the initial value problem [4]:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \mathbf{a}_Q = \frac{1}{m}\mathbf{F}(\mathbf{r},\dot{\mathbf{r}},t),\ \mathbf{r}(t_0)=\mathbf{r}_0,\ \dot{\mathbf{r}}(t_0)=\mathbf{v}_0, \tag{1}$$

where $t_0 \geq 0$ is the initial moment of time, $\boldsymbol{\omega}=\boldsymbol{\omega}(t), t\geq t_0$ represents the instantaneous angular velocity of the non-inertial reference frame, $\mathbf{a}_Q=\mathbf{a}_Q(t), t\geq t_0$ represents the acceleration of the origin of the noninertial reference frame with respect to an inertial reference frame and $\mathbf{F}(\mathbf{r},\dot{\mathbf{r}},t)$ is the force that acts on the particle that has the mass *m*..
In the situation when the force is central and it depends only on the position vector magnitude, i.e. $\mathbf{F}=\mathbf{F}(r)$, then the initial value problem (1) may be written as:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r} + f(r)\mathbf{r} + \mathbf{a}_Q = \mathbf{0},\ \mathbf{r}(t_0)=\mathbf{r}_0,\ \dot{\mathbf{r}}(t_0)=\mathbf{v}_0, \tag{2}$$

where f is a continuous real valued map, $f = -\frac{1}{m}\mathbf{F}$, depending only on the magnitude of the position vector, $f = f(r)$. The vector valued function $\boldsymbol{\omega}$ is supposed to be known. The problem is to determine the law of motion $\mathbf{r} = \mathbf{r}(t)$, which is the solution to the initial value problem (1) or, in the particular case, (2).

3. A TENSOR INSTRUMENT

In this Section we present the tensor instrument that allows reducing the problem of the motion in a non-inertial reference frame to the problem of the motion in an inertial reference frame.

The key element of the method is represented by the Darboux equation, which has as solution the proper orthogonal tensor valued map that models the rotation with a known specific instantaneous angular velocity [5,6].

Let $\boldsymbol{\omega} = \boldsymbol{\omega}(t), t \geq t_0$ be a continuous vector valued map and let $\tilde{\boldsymbol{\omega}}$ be the associated skew-symmetric tensor valued function. Tensor $\tilde{\boldsymbol{\omega}}$ is also known as the "cross-product tensor", since:

$$\tilde{\boldsymbol{\omega}}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}, (\forall)\mathbf{x} \in \mathbf{V}_3, \qquad \textbf{(3)}$$

where $\mathbf{V}_3$ denotes the three-dimensional linear space of free vectors. This property might also be extended to the set of vector functions of real variable $\mathbf{V}_3^{\mathbb{R}}$. The matrix correspondence between tensor $\tilde{\boldsymbol{\omega}}$ and vector $\boldsymbol{\omega}$, in a right handed orthonormate base, is given by [7]:

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \Leftrightarrow \tilde{\boldsymbol{\omega}} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \qquad \textbf{(4)}$$

One particular form of the Darboux equation is the initial value problem:

$$\mathbf{Q} = \mathbf{Q}\tilde{\boldsymbol{\omega}}, \mathbf{Q}(t_0) = \mathbf{I}_3, \qquad \textbf{(5)}$$

where $\mathbf{I}_3$ represents the second order unit tensor.

The solution to the initial value problem (5) will be denoted by $\mathbf{F}_{\boldsymbol{\omega}}$. This tensor valued function might be treated as an operator on the set of vector valued functions of real variable, $\mathbf{F}_{\boldsymbol{\omega}} : \mathbf{V}_3^{\mathbb{R}} \to \mathbf{V}_3^{\mathbb{R}}$.

The properties of this tensor valued function of real variable are listed below; their proof may be found in Ref. [6]

1. $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a} \cdot \mathbf{F}_{\boldsymbol{\omega}}\mathbf{b} = \mathbf{a} \cdot \mathbf{b}, (\forall)\mathbf{a}, \mathbf{b} \in V_3^{\mathbb{R}}$ (it preserves the dot product).
2. $|\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}| = |\mathbf{a}|, (\forall)\mathbf{a} \in V_3^{\mathbb{R}}$ (it is an isometry).
3. $\mathbf{F}_{\boldsymbol{\omega}}\mathbf{a} \times \mathbf{F}_{\boldsymbol{\omega}}\mathbf{b} = \mathbf{a} \times \mathbf{b}, (\forall)\mathbf{a}, \mathbf{b} \in V_3^{\mathbb{R}}$ (it preserves the cross product).
4. $\frac{d}{dt}\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r} = \mathbf{F}_{\boldsymbol{\omega}}(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}), (\forall)\mathbf{r} \in V_3^{\mathbb{R}}$.
5. $\frac{d^2}{dt^2}\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r} = \mathbf{F}_{\boldsymbol{\omega}}\left[\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r}\right], (\forall)\mathbf{r} \in V_3^{\mathbb{R}}$.
6. $\left.\frac{d}{dt}\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}\right|_{t=t_0} = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0, (\forall)\mathbf{r} \in V_3^{\mathbb{R}}$, where $\mathbf{v}_0 = \dot{\mathbf{r}}(t_0), \boldsymbol{\omega}_0 = \boldsymbol{\omega}(t_0)$

The tensor function of real variable $\mathbf{F}_{\boldsymbol{\omega}}$ is invertible, since it is proper orthogonal, and its inverse is its transpose. We denote:

$$\mathbf{R}_{-\boldsymbol{\omega}} = \mathbf{F}_{\boldsymbol{\omega}}^T \qquad \textbf{(6)}$$

The tensor function $\mathbf{R}_{-\boldsymbol{\omega}}$ models the rotation with instantaneous angular velocity $-\boldsymbol{\omega}$. It is the solution to an initial value problem similar to Eq. (5):

$$\dot{\mathbf{R}}_{-\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{R}_{-\boldsymbol{\omega}} = \mathbf{0}_3, \mathbf{R}_{-\boldsymbol{\omega}}(t_0) = \mathbf{I}_3 \qquad \textbf{(7)}$$

In the situation when the instantaneous angular velocity $-\boldsymbol{\omega}$ has a fixed direction, i.e.

$$\boldsymbol{\omega} = \omega\, \mathbf{u}, \omega = \omega(t) \qquad \textbf{(8)}$$

with $\mathbf{u}$ a constant unit vector, then we may explicitly write the expression of the tensor valued function $\mathbf{R}_{-\boldsymbol{\omega}}$ with the help of a Rodrigues-like formula (see Ref. [7]):

$$\mathbf{R}_{-\boldsymbol{\omega}} = \mathbf{I}_3 - \sin\left(\int_{t_0}^{t} \omega(s)\, ds\right)\tilde{\mathbf{u}} + \left[1 - \cos\left(\int_{t_0}^{t} \omega(s)\, ds\right)\right]\tilde{\mathbf{u}}^2 \qquad \textbf{(9)}$$

The authors of the present paper have proved that the Darboux equation (5) has closed form solutions in a much larger number of cases. For instance, when the angular velocity $-\boldsymbol{\omega}$ has uniform precession, i.e. there exists a proper orthogonal tensor valued function of real variable $\mathbf{R}_1$ such as:

$$\begin{cases} -\boldsymbol{\omega} = \mathbf{R}_1\boldsymbol{\omega}_0 \\ \mathbf{R}_1 = \mathbf{I}_3 + \sin\left[\omega_1\left(t-t_0\right)\right]\tilde{\mathbf{u}}_1 + \left[1-\cos\left[\omega_1\left(t-t_0\right)\right]\right]\tilde{\mathbf{u}}_1^2 \end{cases} \quad \textbf{(10)}$$

where $\boldsymbol{\omega}_0$ is a constant vector, $\mathbf{u}_1$ is a constant unit vector and ω_1 is a real number, then the solution to the initial value problem (9), which offers the rotation tensor function associated to the instantaneous angular velocity $-\boldsymbol{\omega}$, is given by (also see Ref. [3]):

$$\begin{aligned} \mathbf{R}_{-\boldsymbol{\omega}} = &\left\{\mathbf{I}_3 + \sin\left[\omega_1\left(t-t_0\right)\right]\tilde{\mathbf{u}}_1 + \left[1-\cos\left[\omega_1\left(t-t_0\right)\right]\right]\tilde{\mathbf{u}}_1^2\right\}\times \\ &\left\{\mathbf{I}_3 + \sin\left[\left(\omega_0-\omega_1\right)\left(t-t_0\right)\right]\left(\tilde{\mathbf{u}}_0-\tilde{\mathbf{u}}_1\right) + \left[1-\cos\left[\left(\omega_0-\omega_1\right)\left(t-t_0\right)\right]\right]\left(\tilde{\mathbf{u}}_0-\tilde{\mathbf{u}}_1\right)^2\right\} \end{aligned} \quad \textbf{(11)}$$

4. A GENERAL APPROACH TO THE MOTION IN A NON-INERTIAL FRAME

By using the tensor instrument introduced in the previous Section, we will offer a general method to deal with the initial value problem (1). First, let us apply operator $\mathbf{F}_{\boldsymbol{\omega}}$ to the initial value problem (1). By using the properties mentioned in the previous Section, we might state that Eq. (1) is equivalent with:

$$\frac{d^2}{dt^2}\left(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}\right) + \mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}_Q = \frac{1}{m}\mathbf{F}_{\boldsymbol{\omega}}\left[\mathbf{F}\left(\mathbf{r},\dot{\mathbf{r}},t\right)\right], \mathbf{r}\left(t_0\right) = \mathbf{r}_0, \dot{\mathbf{r}}\left(t_0\right) = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0, \quad \textbf{(12)}$$

The form of Eq. (12) suggests the introduction of a change of variable in Eq. (1):

$$\boldsymbol{\rho}\left(t\right) := \left(\mathbf{F}_{\boldsymbol{\omega}}\mathbf{r}\right)\left(t\right), t \geq t_0. \quad \textbf{(13)}$$

We now may state the main result of this paper.

Theorem 1 *The solution to the initial value problem (1) is obtained by applying the tensor operator* $\mathbf{R}_{-\boldsymbol{\omega}}$ *to the solution to the initial value problem:*

$$\ddot{\boldsymbol{\rho}} + \mathbf{F}_{\boldsymbol{\omega}}\mathbf{a}_Q = \frac{1}{m}\mathbf{F}'\left(\boldsymbol{\rho},\dot{\boldsymbol{\rho}},t\right), \boldsymbol{\rho}\left(t_0\right) = \mathbf{r}_0, \dot{\boldsymbol{\rho}}\left(t_0\right) = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0, \quad \textbf{(14)}$$

where $\mathbf{F}'\left(\boldsymbol{\rho},\dot{\boldsymbol{\rho}},t\right)$ *is linked to* $\mathbf{F}\left(\mathbf{r},\dot{\mathbf{r}},t\right)$ *by:*

$$\mathbf{F}'\left(\boldsymbol{\rho},\dot{\boldsymbol{\rho}},t\right) = \mathbf{F}_{\boldsymbol{\omega}}\left\{\mathbf{F}\left[\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\rho}, \mathbf{R}_{-\boldsymbol{\omega}}\dot{\boldsymbol{\rho}} - \boldsymbol{\omega}\times\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\rho}, t\right]\right\} \quad \textbf{(15)}$$

Proof. The form of the term in the left of the differential equation (14) is deduced from Eqs. (12) and (13). The initial conditions of the initial value problem (14) are deduced from the property (4.) of the tensor operator $\mathbf{F}_{\boldsymbol{\omega}}$. We have only to justify the existence of the term $\mathbf{R}_{-\boldsymbol{\omega}}\dot{\boldsymbol{\rho}} - \boldsymbol{\omega}\times\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\rho}$ in the expression (15) of the force.
From Eq. (13) and from the property (4) of the tensor operator $\mathbf{F}_{\boldsymbol{\omega}}$ it follows that:

$$\dot{\boldsymbol{\rho}} = \mathbf{F}_{\boldsymbol{\omega}}\left(\dot{\mathbf{r}} + \boldsymbol{\omega}\times\mathbf{r}\right) \quad \textbf{(16)}$$

By using Eqs. (6) and (13), we deduce:

$$\dot{\mathbf{r}} = \mathbf{R}_{-\boldsymbol{\omega}}\dot{\boldsymbol{\rho}} - \boldsymbol{\omega}\times\mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\rho}; \quad \textbf{(17)}$$

this finalizes the proof. □

Remark 1 ***Theorem 1*** *offers a method to study the motion of a mass particle with respect to a non-inertial reference frame. the original initial value problem that models the motion is transform, by an adequate change of variable, into a simpler model. This method is useful to be use din multiple problems related to the non-inertial motion: Kepler's problem in rotating reference frames, the Foucault Pendulum, relative orbital motion..*

5. MOTION IN A CENTRAL FORCE FIELD WITH RESPECT TO A ROTATING FRAME

This type of motion refers to multiple Theoretical Mechanics and Astrodynamics problems. The initial value problem is derived from Eq. (2) by eliminating the transport acceleration $\mathbf{a}_Q$:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega}\times\dot{\mathbf{r}} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{r}\right) + \dot{\boldsymbol{\omega}}\times\mathbf{r} + f\left(r\right)\mathbf{r} = \mathbf{0}, \mathbf{r}\left(t_0\right) = \mathbf{r}_0, \dot{\mathbf{r}}\left(t_0\right) = \mathbf{v}_0, \quad \textbf{(18)}$$

By applying **Theorem 1**, it follows that the solution to the initial value problem (18) is obtained by applying the tensor operator $\mathbf{R}_{-\boldsymbol{\omega}}$ to the solution to the initial value problem [5,6]:

$$\ddot{\boldsymbol{\rho}} + f\left(\rho\right)\boldsymbol{\rho} = \mathbf{0}, \boldsymbol{\rho}\left(t_0\right) = \mathbf{r}_0, \dot{\boldsymbol{\rho}}\left(t_0\right) = \mathbf{v}_0 + \boldsymbol{\omega}_0\times\mathbf{r}_0, \quad \textbf{(19)}$$

This statement leads immediately to a very interesting geometrical interpretation of the motion in rotating reference frame in the presence of a central force field: the motion takes place in a plane, as if it was described by the initial value problem (19). This plane is formed at the initial moment of time t_0 and it rotates with angular velocity $-\boldsymbol{\omega}$.

One of the classic ways of dealing with the initial value problem (19) is by using polar coordinates in the plane described above. If one chooses the polar axis such as its orientation coincides with that of $\mathbf{r}_0$ and he chooses the increasing sense of the polar angle θ having the orientation given by the initial angular momentum of the planar motion,

$$\mathbf{h}_0 = \mathbf{r}_0 \times (\mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0,) \quad \textbf{(20)}$$

then the algorithm for solving the initial value problem (18) has the following steps:

1. Consider the system of scalar differential equations for ρ and θ:

$$\begin{cases} \ddot{\rho} - \rho\dot{\theta}^2 + f(r) = 0 \\ \rho^2\dot{\theta} = h_0 \end{cases}, \text{ with the initial conditions } \rho(t_0) = r_0, \dot{\rho}(t_0) = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0}, \theta(t_0) = 0 \quad \textbf{(21)}$$

2. Solve the initial value problem for the magnitude of the position vector *r*:

$$\ddot{\rho} - \frac{h_0^2}{\rho^3} + f(r) = 0, \rho(t_0) = r_0, \dot{\rho}(t_0) = \frac{\mathbf{r}_0 \cdot \mathbf{v}_0}{r_0}, \quad \textbf{(22)}$$

   where h_0 is the magnitude of the vector defined in Eq. (20).

3. Obtain the expression of the real function θ from the second equation in (21).
4. Express vector $\boldsymbol{\rho}$, the solution to the initial value problem (19).
5. Determine vector **r**, the solution to the initial value problem (18), by using:

$$\mathbf{r} = \mathbf{R}_{-\boldsymbol{\omega}}\boldsymbol{\rho} \quad \textbf{(23)}$$

Note that the above algorithm may be considered just a suggestion to approach the initial value problem (19). If vector $\boldsymbol{\rho}$ is obtained by whatever means, only the last two steps of the above algorithm must be executed, which in fact transfer the problem from the inertial frame back to the rotating frame.
The next subsections present some important applications to this method.

5.1. The Kepler problem in rotating reference frames

The initial value problem that models the motion has the form:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \frac{\mu}{r^3}\mathbf{r} = \mathbf{0}, \mathbf{r}(t_0) = \mathbf{r}_0, \dot{\mathbf{r}}(t_0) = \mathbf{v}_0, \quad \textbf{(24)}$$

where μ is the gravitational parameter of the attraction center. Following **Theorem 1**, the solution to the initial value problem (24) is obtained by applying the tensor operator $\mathbf{R}_{-\boldsymbol{\omega}}$ to the solution to the initial value problem:

$$\ddot{\boldsymbol{\rho}} + \frac{\mu}{\rho^3}\boldsymbol{\rho} = \mathbf{0}, \boldsymbol{\rho}(t_0) = \mathbf{r}_0, \dot{\boldsymbol{\rho}}(t_0) = \mathbf{v}_0 + \boldsymbol{\omega}_0 \times \mathbf{r}_0, \quad \textbf{(25)}$$

Note that Eq. (25) models the Keplerian motion with respect to an inertial frame, and its solution is considered to be known. A comprehensive approach to the Kepler problem in rotating reference frames may be found in [5].

5.2. The Foucault Pendulum

The initial value problem that models the Foucault Pendulum motion is:

$$\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega_*^2\mathbf{r} = \mathbf{0}, \mathbf{r}(t_0) = \mathbf{r}_0, \dot{\mathbf{r}}(t_0) = \mathbf{v}_0, \quad \textbf{(26)}$$

where ω_* is the pulsation of the pendulum (it depends on its length and the gravitational acceleration at the experiment place). Here vector $\boldsymbol{\omega}$ is constant and it models the Earth rotation around its pole axis. Following **Theorem 1**, the solution to the initial value problem (26) is obtained by applying the tensor operator $\mathbf{R}_{-\boldsymbol{\omega}}$ to the solution to the initial value problem:

$$\ddot{\boldsymbol{\rho}} + \omega_*^2\boldsymbol{\rho} = \mathbf{0}, \boldsymbol{\rho}(t_0) = \mathbf{r}_0, \dot{\boldsymbol{\rho}}(t_0) = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_0, \quad \textbf{(27)}$$

The amazing quality of this method is that in the particular case of the Foucault Pendulum motion (which in most Theoretical Mechanics textbooks is considered to lack a closed form solution) offers a simple way to solve it by reducing it to the elementary problem of a harmonic oscillator described by the initial value problem (27). The solution to the initial value problem (27) is:

$$\boldsymbol{\rho}(t) = \mathbf{r}_0 \cos\left[\omega_*(t - t_0)\right] + \frac{\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_0}{\omega_*}\sin\left[\omega_*(t - t_0)\right], \quad \textbf{(28)}$$

so the solution to the initial value problem (26) is:

$$\mathbf{r}(t) = \cos\left[\omega_*(t - t_0)\right](\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_0) + \sin\left[\omega_*(t - t_0)\right]\left(\mathbf{R}_{-\boldsymbol{\omega}}\frac{\mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_0}{\omega_*}\right) \quad \textbf{(29)}$$

By taking into account that the angular velocity $\boldsymbol{\omega}$ of the rotating frame is constant, then the tensor operator $\mathbf{R}_{-\boldsymbol{\omega}}$ is expressed like:

$$\mathbf{R}_{-\boldsymbol{\omega}} = \mathbf{I}_3 - \sin\left[\omega(t - t_0)\right]\tilde{\mathbf{u}} + \left\{1 - \cos\left[\omega(t - t_0)\right]\right\}\tilde{\mathbf{u}}^2 \quad \textbf{(30)}$$

where **u** is the unit vector associated to vector $\boldsymbol{\omega}$. The explicit solution to the Foucault Pendulum motion described by the initial value problem (26) may be written as:

$$\mathbf{r}(t)=\cos\left[\omega_*(t-t_0)\right]\left\{\mathbf{r}_0-\sin\left[\omega(t-t_0)\right]\tilde{\mathbf{u}}\mathbf{r}_0+\left\{1-\cos\left[\omega(t-t_0)\right]\right\}\tilde{\mathbf{u}}^2\mathbf{r}_0\right\}+$$
$$+\sin\left[\omega_*(t-t_0)\right]\left\{\frac{\mathbf{v}_0+\boldsymbol{\omega}\times\mathbf{r}_0}{\omega_*}-\sin\left[\omega(t-t_0)\right]\tilde{\mathbf{u}}\frac{\mathbf{v}_0+\boldsymbol{\omega}\times\mathbf{r}_0}{\omega_*}+\left\{1-\cos\left[\omega(t-t_0)\right]\right\}\tilde{\mathbf{u}}^2\frac{\mathbf{v}_0+\boldsymbol{\omega}\times\mathbf{r}_0}{\omega_*}\right\} \quad \textbf{(31)}$$

A comprehensive approach of the Foucault pendulum – like motion may be found in […].

5.3. The relative motion of satellites

Consider two spacecrafts orbiting around the same attraction center under the influence of an unperturbed gravitational field. One of them will be referred as the Chief, and the other one the Deputy. The reference frame where the motion of the Deputy satellite is studied is named LVLH (Local-Vertical-Local-Horizontal) and its axes are oriented as it follows: the *Ox* axis has the same orientation as the inertial position vector of the Chief satellite, the *Oz* axis has the same direction and sense as the angular momentum of the inertial orbit of the Chief satellite; the *Oy* axis completes a right handed orthogonal frame. This frame has the origin in the Chief satellite center of mass. Denote by $-\mathbf{r}_C$ the position vector of the attraction center with respect to the LVLH frame and denote by $\boldsymbol{\omega}$ the angular velocity of the LVLH frame. The initial value problem that models the motion of the Deputy satellite with respect to LVLH is:

$$\ddot{\mathbf{r}}+2\boldsymbol{\omega}\times\dot{\mathbf{r}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\dot{\boldsymbol{\omega}}\times\mathbf{r}+\frac{\mu}{\left|\mathbf{r}+\mathbf{r}_C\right|^3}(\mathbf{r}+\mathbf{r}_C)-\frac{\mu}{r_C^3}\mathbf{r}_C=\mathbf{0},\ \mathbf{r}(t_0)=\Delta\mathbf{r},\ \dot{\mathbf{r}}(t_0)=\Delta\mathbf{v} \quad \textbf{(32)}$$

The problem is of great interest for this approach, since it is the same type as Eq. (1). The force in the right hand term depends only of time and the position vector.
A direct approach by using the tensor instrument introduced in Section 4 does not lead to satisfactory results, but there exist a very ingenuous way to solve the problem. The initial value problem (32) may be seen as the difference between two initial value problems of the same type, having different initial conditions. One is:

$$\ddot{\mathbf{r}}+2\boldsymbol{\omega}\times\dot{\mathbf{r}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\dot{\boldsymbol{\omega}}\times\mathbf{r}+\frac{\mu}{r^3}\mathbf{r}=\mathbf{0},\ \mathbf{r}(t_0)=\mathbf{r}_0+\Delta\mathbf{r},\ \dot{\mathbf{r}}(t_0)=\frac{\mathbf{v}_0\cdot\mathbf{r}_0}{r_0^2}\mathbf{r}_0+\Delta\mathbf{v} \quad \textbf{(33)}$$

and the other one is:

$$\ddot{\mathbf{r}}+2\boldsymbol{\omega}\times\dot{\mathbf{r}}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})+\dot{\boldsymbol{\omega}}\times\mathbf{r}+\frac{\mu}{r^3}\mathbf{r}=\mathbf{0},\ \mathbf{r}(t_0)=\mathbf{r}_0,\ \dot{\mathbf{r}}(t_0)=\frac{\mathbf{v}_0\cdot\mathbf{r}_0}{r_0^2}\mathbf{r}_0 \quad \textbf{(34)}$$

where vector $-\mathbf{r}_0$ represents the position vector of the attraction center with respect to LVLH at the initial moment of time t_0 and $\mathbf{v}_0$ represents the initial velocity of the Chief satellite with respect to an Earth centered inertial frame. The solution to Eq. (34) is $\mathbf{r}_C$, and it models a rectilinear motion. From a geometrical point of view, it models the motion of the Chief satellite with respect to a LVLH frame originated in the attraction center.
The inertial orbit of the Chief satellite is associated to a Keplerian motion, so vector $\mathbf{r}_C$ might be expressed as:

$$\mathbf{r}_C=\frac{p_C}{1+e_C\cos f_C}\frac{\mathbf{r}_0}{r_0}, \quad \textbf{(35)}$$

where p_C is the semilatus rectum, e_C is the eccentricity and $f_C=f_C(t)$ is the true anomaly, all associated to the inertial orbit of the Chief satellite. Also denote by $\mathbf{h}_C$ the specific angular momentum of the inertial orbit of the Chief satellite. The angular velocity of the LVLH reference frame may be expressed as:

$$\boldsymbol{\omega}=\frac{\mathbf{h}_C}{p_C^2}(1+e_C\cos f_C)^2 \quad \textbf{(35)}$$

By applying the tensor instrument presented before in this paper, we may state that the solution to the initial value problem (32) is:

$$\mathbf{r}=\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_*-\frac{p_C}{1+e_C\cos f_C}\frac{\mathbf{r}_0}{r_0}, \quad \textbf{(36)}$$

where $\mathbf{r}_*$ is the solution to the initial value problem:

$$\ddot{\mathbf{r}}+\frac{\mu}{r^3}\mathbf{r}=\mathbf{0},\ \mathbf{r}(t_0)=\mathbf{r}_0+\Delta\mathbf{r},\ \dot{\mathbf{r}}(t_0)=\mathbf{v}_0+\Delta\mathbf{v}+\boldsymbol{\omega}_0\times\Delta\mathbf{r}, \quad \textbf{(37)}$$

The result expressed in Eq. (36) allows us to determine the relative velocity of the Deputy satellite with respect to LVLH. It is:

$$\mathbf{v}=\mathbf{R}_{-\boldsymbol{\omega}}\dot{\mathbf{r}}_*-\tilde{\boldsymbol{\omega}}\mathbf{R}_{-\boldsymbol{\omega}}\mathbf{r}_*-\frac{e_C h_C\sin f_C}{p_C}\frac{\mathbf{r}_0}{r_0} \quad \textbf{(38)}$$

A comprehensive analysis of the relative orbital motion by using this procedure may be found in [8,9,10]. The solution for the relative orbital motion presented here is a generalization of several particular solutions for the situations when the reference trajectory is circular [11] and when the reference trajectory is elliptic [12-14]

6. CONCLUSIONS

The paper presents a general method for the study of the motion in a non-inertial reference frame. This method is based on proper orthogonal and skew-symmetric tensor valued functions, which are introduced by the Darboux equation. The case of the motion in a central force field with respect to a rotating reference frame is studied and an algorithm for determining the law of motion in this situation is proposed. Three applications are presented: the Kepler problem in rotating reference frames, the Foucault Pendulum and the relative motion of satellites. The solution offered for the relative orbital motion is a generalization of the solutions offered by Clohessy and Wiltshire for the circular reference trajectory and by Lawden and Tschauner – Hempel for the elliptic reference trajectory.

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