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ON THE STATE OF STRESS AND STRAIN AT DISKS IN THERMAL REGIME, IN THE PRESENCE OF RESIDUAL STRESSES PART II – DISKS, CALCULATION RELATIONS AND EXPERIMENTAL DETERMINATION

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***Abstract**: The present paper presents the theoretical method for determination of the deformations and the loss of stability of the disk stressed by an axial – symmetric thermal field, variable according to disk radius and thickness, superposed with a field of membrane tensions given by the rotational motion. The experimental results confirm the theoretical hypothesis. This paper also presents the tension and deformation state of the disks being in a non – stationary field of temperature. The study is done until the plastic deformations occur.*
***Keywords**: disk, thermal field, stress, membrane tension, stability, non – stationary field*

1. INTRODUCTION. PROBLEM FORMULATION

From the equilibrium element of disk and using the hypothesis that the load is zero, between tensions exists the following relation:

$$\frac{d(r\sigma r)}{dr}-\sigma_{\varphi}=0 \tag{1}$$

where σ_z=0. It is assumed that the disk stress given by the thermal field is realized inside of the elastic domain. There are to be used the known relations between tensions σ, deformations ε and temperature T as well as the relations between deformations u and specific deformations ε. We know that:

$$\begin{cases}\varepsilon_{re}=\dfrac{1}{E}(\sigma_r-\upsilon\sigma_{\varphi})+\alpha T\\ \varepsilon_{\varphi r}=\dfrac{1}{E}(\sigma_{\varphi}-\upsilon\sigma_r)+\alpha T\\ \varepsilon_{ze}=-\dfrac{\upsilon}{E}(\sigma_r+\sigma_{\varphi})+\alpha T\end{cases}$$

The next calculation relation is obtained:

$$\frac{d^2u_e}{dr^2}+\frac{1}{r}*\frac{du_e}{dr}-\frac{1}{r^2}-\alpha(1+\upsilon)*\frac{dT}{dr}=0 \tag{2}$$

with the solution of

$$u_e=A\frac{1}{r}+Br+\alpha\frac{1+\upsilon}{r}\int rTdr \tag{3}$$

Finally there are obtained the calculation relations for tensions:

$$\begin{cases}\sigma_{re} = \frac{1}{1-\upsilon^2}[-A\frac{1}{r^2}(1-\upsilon)+B(1+\upsilon)- \\ \alpha(1-\upsilon^2)\int rTdr]; \\ \sigma_{\varphi e} = \frac{1}{1-\upsilon^2}[A\frac{1}{r^2}(1-\upsilon)+B(1+\upsilon)+ \\ \alpha(1-\upsilon^2)(\frac{1}{r^2}\int sTdr - T)]\end{cases} \tag{4}$$

The integration constants A and B will be determined with the mean of limit conditions, respectively with the mean of continuity conditions at the limits between the elastic domain and plastic domain.
In the plastic domain there should be satisfied the plasticity criteria as well.
In the case of ring – shaped disks the unified fields criterion is adequate. Essentially this criterion is based on the hypothesis which stipulates that the plastic yielding is checked by the action and combination of two shear stresses.
The mathematical expression of the unified fields criterion is given by the following equations:

$$\begin{aligned}&\tau_{13} + b\tau_{12} = c, \quad when \quad \tau_{12} \geq \tau_{23} \\ &\tau_{13} + b\tau_{23} = c, \quad when \quad \tau_{23} \geq \tau_{12}\end{aligned} \tag{4'}$$

where b is a weight coefficient which shows the influence of the shear stresses τ_{12} or τ_{23} on the material and c is a material coefficient.
The equations are valid in the following conditions:
→ $\sigma_1 \geq \sigma_2 \geq \sigma_3$ are main stresses;
→ (σ_1, σ_2); (σ_2, σ_3) and (σ_3, σ_1) ar shear planes;

$$\tau_{12} = \frac{\sigma_1 - \sigma_2}{2}; \tau_{23} = \frac{\sigma_2 - \sigma_3}{2}\ and$$
$$\tau_{13} = \frac{\sigma_1 - \sigma_3}{2}$$

represents the shear stresses.

With this explanation, Eqs. (4) become:

$$\begin{cases}\sigma_1 - \frac{1}{1+b}(b\sigma_2 + \sigma_3) = \sigma_r \quad for \frac{\sigma_1 + \sigma_3}{2} \geq \sigma_2; \\ \frac{1}{1+b}(\sigma_1 + b\sigma_2) - \sigma_3 = \sigma_0 \quad for \frac{\sigma_1 + \sigma_3}{2} < \sigma_2\end{cases} \tag{5}$$

where σ_r is the tensile rupture limit of the material.
Eqs. (5) have a general feature. One mention that:
- for b = 0 one come to the TRESCA criterion;
- for b = 0.5 one come to the MISES criterion;
- for b = 1 one come to the unified fields criterion.

In hypothesis of Huber, Mises, Hencky plasticity criteria, there should be fulfilled the condition expressed through relation:

$$\sigma_{\varphi p}^2 - \sigma_{\varphi p}\sigma_{rp} + \sigma_{rp}^2 = \sigma_c \tag{6}$$

where σ_c – flowing limit of the material at the uniaxial test of lengthening. From the above:

$$\sigma_{\varphi p} = \frac{\sigma_{rp}}{2} \pm \sqrt{\sigma_c^2 - \frac{3}{4}\sigma_{rp}^2} \tag{7}$$

On the inner edge of the ring-shaped disk, from where actually the plasticity process starts, in the case they will be treated, $\sigma_{rp}(a)=0$. In front of the square root, the minus sign will be chosen at heating because of the negative tangential tension and at cooling there will be chosen the positive sign. Utilizing index1 for heating and index 2 for cooling it is obtained:

$$\frac{d\sigma_{rp1}}{dr} = -\frac{\sigma_{rp1}}{2} - \frac{1}{r}\sqrt{\sigma_c^2 - \frac{3}{4}\sigma_{rp1}^2} \tag{8}$$

This equation may be solved, in the case of a variable flowing limit $\sigma_c(r)$, by the mean of Runge-Kutta method, and the obtaining tensions will be approximated through parabola. For the mean flowing limit, constant:$\sigma_c=\sigma_m$, equation (8) may be integrated and therefore results:

$$\sigma_{rp1} = \frac{-2\sigma_{m1}}{\sqrt{3}}\sin\left(\upsilon - \frac{2}{3}\pi\right) \tag{9}$$

$\sigma_{\varphi p1}$ value results from plasticity criteria:

$$\sigma_{\varphi p1} = \frac{2\sigma_{m1}}{\sqrt{3}}\sin\left(\upsilon - \frac{2}{3}\pi\right) \tag{10}$$

In the same way, for he root case, the radial and tangential normal tensions will have their values given by the next relations:

$$\sigma_{rp2} = \frac{2\sigma_{m2}}{\sqrt{3}}\sin\left(\upsilon - \frac{2}{3}\pi\right) \quad ; \quad \sigma_{\varphi p2} = -\frac{2\sigma_{m2}}{\sqrt{3}}\sin\left(\upsilon - \frac{2}{3}\pi\right) \tag{11}$$

In several cases is advantage the utilization of Tresca's plastic condition. Usually deviations that result against the obtained results using the Mises's condition are insignificantly. The main advantage of Tresca's plastic criteria consists of the fact hat it offers relations much simplified and thus mathematical operations may be used.

In this case the plasticity criteria may be wrote as follow:

$$\begin{cases} (\sigma_{rp} - \sigma_{\varphi p})^2 - \sigma_c^2 = \phi_z \\ (\sigma_{\varphi p} - \sigma_{zp})^2 - \sigma_c^2 = \phi_r \\ (\sigma_{zp} - \sigma_{rp})^2 - \sigma_c^2 = \phi_\varphi \end{cases} \tag{12}$$

flowing appears when one of the three plasticity forces disappears. In the studied cases, $\sigma_{\varphi p}$ and σ_{rp} have the same sign and thus because σ_{zp}=0, plasticity criteria takes the following form:

$$\begin{cases} \phi_r = 0 \\ \sigma_{\varphi p} = \pm\sigma_c \end{cases} \tag{13}$$

The negative sign is valid for heating and the positive sign is valid for cooling the ring-shaped disk. Therefore:

$$\frac{d(\sigma_{rp} r)}{dr} = \pm\sigma_c \tag{14}$$

Taking in consideration the limit conditions $\sigma_{rp}(a)=0$, after the integration operation there will be obtained:

$$\sigma_{rp} = \pm\frac{1}{r}\int_a^r \sigma_c(\bar{r})d\bar{r} \tag{15}$$

Up to now, there have been determined the relations regarding to the elastic domain and plastic one. Next, the coupling of the two domains take place, through some binding relations. It may be assumed that the plastic domain is limited by the inner radius of the ring-shaped disk "a" and a radius of an elastic-plastic limit R_1. Therefore:

$a \le r \le R_1$

The above relation is valid at heating. In the case of cooling there will be a similar relation, but having another radius limit:

$a \le r \le R_2$

Because the temperature may be lowered only to the value of ambient temperature; always:

$R_2 < R_1$

The integration constants A and B as well as the limit radius R result from the following reason: radial tension disappears at the outer radius level:

$$\sigma_{re}(b) = 0 \tag{16}$$

From equilibrium reasons, radial tensions from the plastic domain and elastic domain should have the same values for the limit radius between domains:

$$\sigma_{rp}(R) = \sigma_{re}(R) \tag{17}$$

The plasticity criteria is satisfied in both domains for limit radius R. Therefore:

$$\sigma_{\varphi p}(R)_e = \sigma_{\varphi e}(R) \tag{18}$$

Next, it is assumed that before heating there were no residual tensions in the plate. Therefore σ_r=0 and σ_φ=0. The hypothesis that there existed residual tensions is also used. For heating the value of final stationary temperature T is introduced. Through an accordingly mathematical development the tensions expressions result:

$$\begin{cases} \sigma_{\varphi e1}(r) = \frac{1}{2}(\sigma_{rp1}(R_1) + \sigma_{\varphi p1}(R_1) + E\alpha T_1 \\ (R_1))(1 + \frac{b^2}{r^2}) - E\alpha\frac{1}{r^2}(\int^b rT_1 dr - \int rT_1 dr) - E\alpha T_1 \end{cases} \qquad \begin{cases} \sigma_{re1}(r) = \frac{1}{2}(\sigma_{rp1}(R_1) + \sigma_{\varphi p1}(R_1) + E\alpha T_1 \\ (R_1))(1 + \frac{b^2}{r^2}) + E\alpha\frac{1}{r^2}(\int_a^b rT_1 dr - \int rT_1 dr) \end{cases} \tag{19}$$

For numerical development there will be utilized for T_1 a thermal stationary field using the approximated relations 15 (from the paper "RING - SHAPED DISKS IN NON – STATIONARY THERMAL FIELD").

Determination of residual tension state is done in the same way but taking in consideration that before beginning the process of cooling there are the following tensions σ_{r1} and $\sigma_{\varphi 1}$. Knowing these values is only important for elastic domain.

Tensions from plastic domain σ_{rp2} and $\sigma_{\varphi p2}$ may be calculated, independently of thermal tensions, for constant flowing limit at ambient temperature $T_2=0$.

In the case of first ratio, the following approximate relation is obtained:

$$\frac{\overline{T}}{T} = e^{-2350\left(\frac{ls}{t}\right)^{1.1}\left(\frac{r}{a}-1\right)}\left[0.0748 + 0.925e^{-1.920\left(\frac{r}{a}-1\right)}\right] \qquad (20)$$

Representations from Fig. 5-6 (from the paper "RING - SHAPED DISKS IN NON – STATIONARY THERMAL FIELD"), had been done based on the following relations:

$$\frac{\overline{T}}{^\circ C} = 30.64 + 2510.7 * e^{-5.75\frac{r}{b}}; \qquad \frac{\overline{T}}{^\circ C} = 45.96 + 3766 * e^{-5.57\frac{r}{b}} \qquad (21)$$

where T_0=400 and 600 C and a/b=1/3. If calculations are made for a/b=1/10 ratio and temperatures T^1_0 equals with 400 and 600 C, the following relations are obtained:

$$\frac{\overline{T}}{^\circ C} = 9.07 + 734 * e^{-6.3\frac{r}{b}}; \qquad \frac{\overline{T}}{^\circ C} = 13.61 + 1101 * e^{-6.3\frac{r}{b}} \qquad (22)$$

The above relations are particularization of relations (15) (from the paper "RING - SHAPED DISKS IN NON – STATIONARY THERMAL FIELD"). We specify that the above relation is an approximated relation of thermal field; in this way a simplification of numerical calculation is obtained, in the way that the tensions state can be determined. It is emphasize that the determined thermal fields using this method differs very little of those considered to be exact (thermal accepted without any reserves).

2. PROBLEM SOLUTION. CONCLUSIONS.

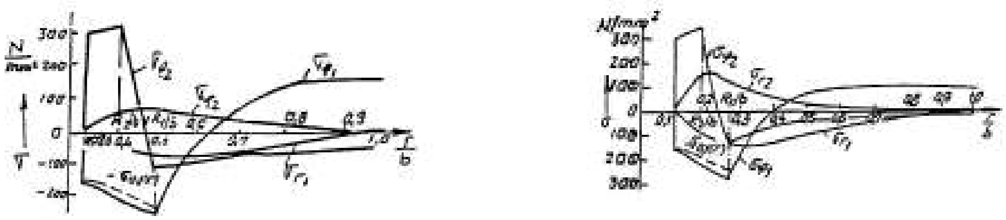
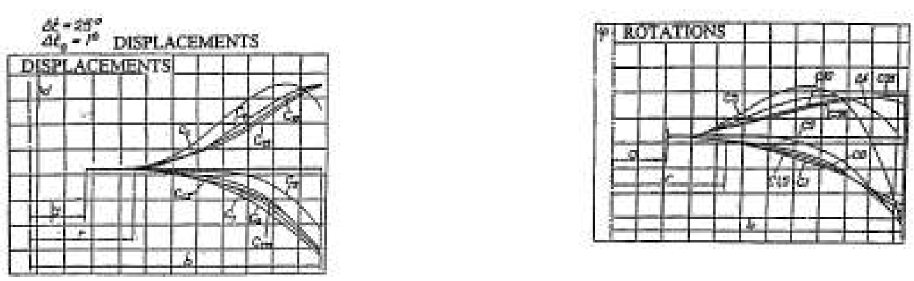


Figure 1

In Fig.1. are presented the residual tensions values for the case of b=1 m, $T_0=600^\circ C$ and ratios of a/b=1/3 and a/b=1/10.



Displacement's *w* variation according to the temperatures' changes on the radius

Rotation's variation according to the temperature's changes on the radius

Figure 2

In Fig. 2 are shown some diagrams, which show the deformation state (rotations and movement) and tension state for thin rotating disks in a thermal axial-symmetric state.

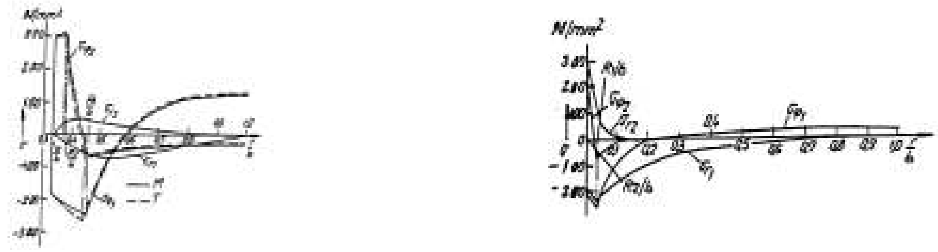


Fig 3

In Fig. 3 is shown the thermal and residual tensions variation for a ring-shaped plate with b=1 m, a/b=1/3 and T_0=400°C. With continuos line were presented the calculated variations using Tresca's criteria, and with dashed line were presented the calculated variations using Mises's criteria. There are no differences for σ_r tensions.

Conclusions are very important and very useful in practice. If curves from Fig.8.b. (from the paper "RING - SHAPED DISKS IN NON – STATIONARY THERMAL FIELD") are compared with those from Fig. 2. and 3. there should be seen great similitude.

This may be explained by the correct way in which the temperature effect for planar plates had been seen and approached.

In conclusion we may say that the elastic stability problems are necessary, their study being as necessary as the static one. In this way it is shown which are the "reserves" of a structure, finishing with a design based on new considerations.

The study of behavior and stability for thin rotating disks in thermal axial-symmetric state, done in the hypothesis of bending stresses as well as membrane stresses, are decoupled (linear theory-first rank calculation), as well as in the hypothesis inter-independence study between this (non-linear behavior-second rank calculation) are necessary. If the disk's thickness is big enough and disks are placed in a non-stationary thermal field, then is taking account of apparition of local plastic deformation, which lead to the apparition of residual tensions (is a natural consequence of dynamic character of thermal processes).

The Unified Fields Criterion can also be named "the uniform yielding criterion" and it is suitable for thin disks. The previous mentioned criterion contains a "family" of cases, which are obtained by variation of parameter *b* between the values 0 and 1.

For the specific cases when b = 0; b = 0.5 and b = 1, the unified fields criterion allows the checking and analysis with maximum efficiency of all structures which are subjected to the TRESCA and MISES criterion. This criterion offers the advantage of allowing the analysing of complex situations, numerical applications which can be solved with help of "linear programming methods".

The disk cannot be perfectly planar; it always has small geometrical flaws. In this situation, the field of membrane tensions produces displacement w perpendicular to the median surface of the disk, even if the field level is bellow the stability loss value. This dependence is not linear.

According to the nature and size of inhomogenies, loss of stability may take place by "axial-symmetrical modes"(with nodal circles) or "axial unsymmetrical modes" (with nodal diameters).

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