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RESPONSE OF DUFFING OSCILLATOR UNDER NARROW-BAND RANDOM EXCITATION

Petre STAN,
Metallurgical High School, Slatina, Romania, e-mail: **petre_stan_marian@yahoo.com**

Abstract: ***Nonlinear, dynamic systems subject to random excitations are frequently met in engineering practice. The source of randomness*** *can vary from surface randomness in vehicle motion and environmental changes, such as earthquakes or wind exciting high rise* ***buildings or wave motions at sea exciting ofshore structures or ships, to electric or acoustic noise exciting mechanical structures. The research goals are, firstly, the computation of stochastic, nonlinear response characteristics (with accuracy and efficiency as important criteria) and, secondly, the investigation and thorough understanding of stochastic, nonlinear response phenomena. The desire to compute response characteristics, such as the power spectral density of the response of these systems, leads to the development of methods that can be used to approximate this response. The excitations, that will be studied, are stationary, Gaussian processes.***

1. INTRODUCTION

We present a method for estimating the power spectral density of the stationary response of oscillator with a nonlinear restoring force under external stochastic wide-band excitation. An equivalent linear system is derived, from which the power spectral density is deduced. The method of the stochastic equivalent linearization is based on the idea that a nonlinear system may be replaced by a linear system by minimizing the mean square error of the two systems. This method has seen the broadest application because of their ability to accurately capture the response statistics over a wide range of response levels while maintaining relatively light computational burden

2. SOLUTION OF EQUATION OF MOTION

Consider the equation of motion

$$m\ddot{\eta}(t)+c\dot{\eta}(t)+k\eta(t)+\alpha k\eta^{3}(t)=F(t)\,. \tag{1}$$

The reduced equation is

$$\ddot{\eta}(t)+2\xi p\dot{\eta}(t)+p^{2}\eta(t)+p^{2}\alpha\eta^{3}(t)=f(t)\,. \tag{2}$$

As a next let us consider excitation described by subsequent correlation function[29]

$$R_F(\tau)=De^{-\lambda\tau}\cos\beta\tau\,, \tag{3}$$

where parameters D>0, λ>0, $\beta\geq 0$.

Power spectral density function [1] of excitation we obtain from the relation:

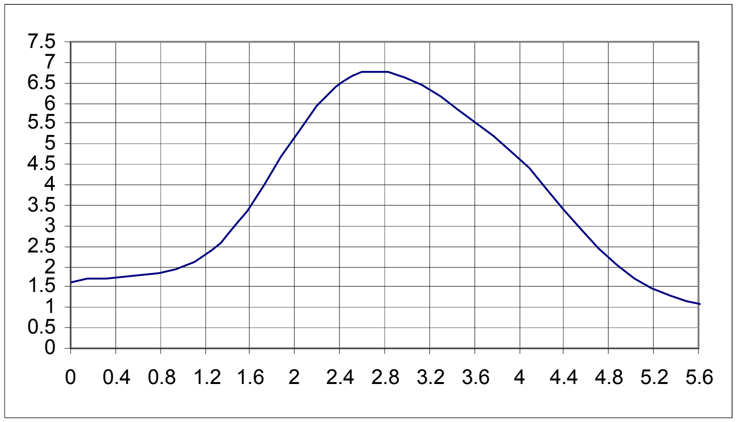
$$S_F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}R_F(\tau)d\tau \tag{4}$$

By substitution of the (3) in the (4) and integration we obtain

$$S_F(\omega)=\frac{D\lambda}{\pi}\frac{\omega^2+\lambda^2+\beta^2}{\left|(i\omega)^2+2\lambda(i\omega)+\lambda^2+\beta^2\right|^2} \tag{5}$$

or

$$S_F(\omega)=\frac{D\lambda}{\pi}\frac{\omega^2+\lambda^2+\beta^2}{\left(\lambda^2+\beta^2-\omega^2\right)^2+4\lambda^2\omega^2}\,. \tag{6}$$



$\frac{\omega}{2\pi}$

Fig. 1. The power spectral density $S_F\ [N^2 \cdot s]$ of excitation for $D = 50\ N^2, \lambda = 1s^{-1}, \beta = 3s^{-1}$.

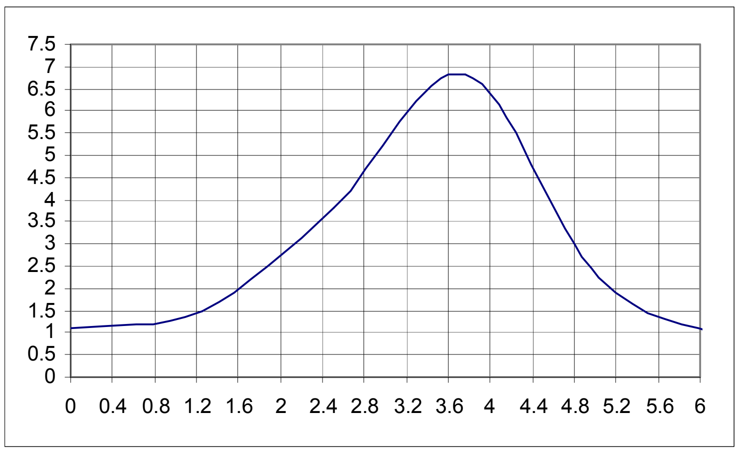


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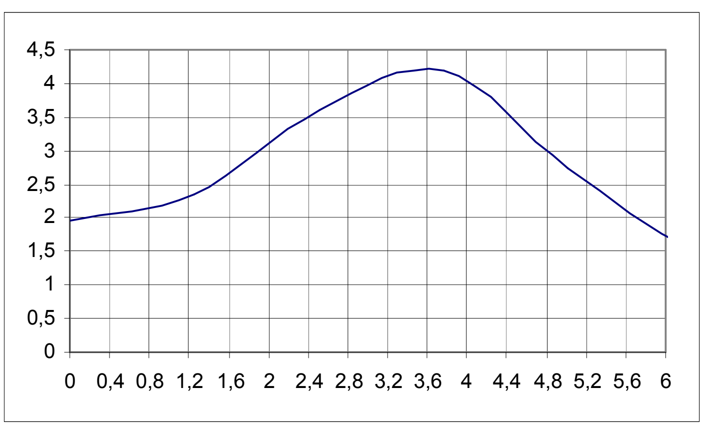
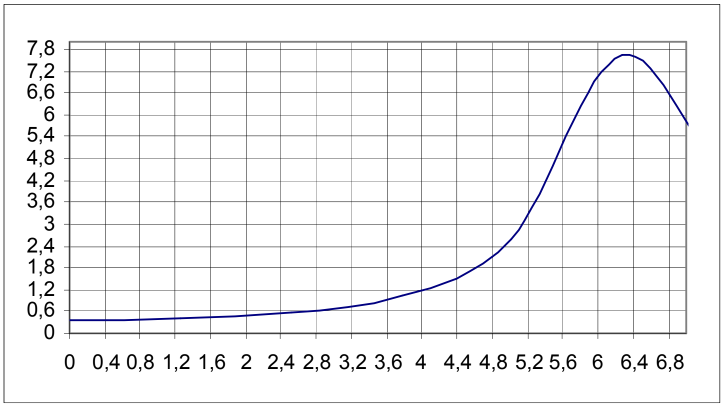
Fig. 2. The power spectral density $S_F\ [N^2 \cdot s]$ of excitation for $D = 45N^2, \lambda = 1s^{-1}, \beta = 3,5s^{-1}$.



$\frac{\omega}{2\pi}$

Fig. 3. The power spectral density $S_F\ [N^2 \cdot s]$ of excitation for $D = 50N^2, \lambda = 2s^{-1}, \beta = 3,5s^{-1}$.



$$\frac{\omega}{2\pi}$$

Fig.4 The power spectral density $S_F\ [N^2 \cdot s]$ of excitation for $D = 50N^2, \lambda = 1s^{-1}, \beta = 6{,}5s^{-1}$.

Power spectral density function of output we can obtain from the relation

$$S_\eta(\omega) = \frac{S_F(\omega)/m^2}{(p_e^2-\omega^2)^2+4\xi^2p^2\omega^2}\,. \tag{7}$$

So we obtain

$$S_\eta(\omega) = \frac{D\lambda(\omega^2+\lambda^2+\beta^2)}{\pi m^2\left\{\left[p^2-\omega^2+3\alpha p^2\sigma^2_\eta\right]^2+4\xi^2p^2\omega^2\right\}\left[\left(\lambda^2+\beta^2-\omega^2\right)^2+4\lambda^2\omega^2\right]}\,. \tag{8}$$

The displacement variance [2] of the single-degree of freedom system under Gaussian white noise excitation can be expressed as,

$$\sigma^2_\eta = R_\eta(0) = \int_{-\infty}^{\infty} S_\eta(\omega)d\omega\,. \tag{9}$$

Substitution of the (8) in the (9) and obtain

$$\sigma^2_\eta = \frac{D\lambda}{m\pi}\int_{-\infty}^{\infty}\frac{(\omega^2+\lambda^2+\beta^2)}{\left\{\left[p^2-\omega^2+3\alpha p^2\sigma^2_\eta\right]^2+4\xi^2p^2\omega^2\right\}\left[\left(\lambda^2+\beta^2-\omega^2\right)^2+4\lambda^2\omega^2\right]}d\omega\,. \tag{10}$$

Integration [3,4] obtain

$$\int_{-\infty}^{\infty}\frac{\omega^2+d}{\left|(i\omega)^2+2\lambda(i\omega)+d\right|^2\left|(i\omega)^2+b_1(i\omega)+b_0\right|^2}d\omega = \frac{\pi(b_o h_1+h_1h_2-h_3)}{b_0(h_1h_2h_3-b_oh_1^2d-h_3^3)}\,, \tag{11}$$

where

$$h_1 = b_1+2\lambda,\quad h_2 = b_0+2\lambda b_1+d,\quad h_3 = 2\lambda b_0+db_1\,. \tag{12}$$

In this case

$$\begin{gathered}h_1 = 2(\xi p+\lambda),\ h_2 = p_e^2+4\lambda\xi p+\lambda^2+\beta^2,\ h_3 = 2\lambda p_e^2+2\xi p(\lambda^2+\beta^2)\\ b_0 = p_e^2 = p^2(1+3\alpha\sigma^2_\eta),\ b_1 = 2\xi p.\end{gathered} \tag{13}$$

$$\sigma^2_\eta = \frac{A}{B}\,, \tag{14}$$

where

$$\begin{aligned}A = {} & D\lambda\sigma^2_\eta\,[12\alpha p^2(\xi p+\lambda)-6\alpha\lambda p^2]+D\lambda\{2p^2(\xi p+\lambda)+\\ & +8\lambda\xi p(\xi p+\lambda)+2(\xi p+\lambda)(\lambda^2+\beta^2)-2\lambda p^2-2\xi p(\lambda^2+\beta^2)\}\end{aligned} \tag{15}$$

$$B=108m\sigma^6_\eta\,\lambda\xi\alpha^3 p^7+36\sigma^4_\eta\,p^4 m\alpha\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+\beta^2)](\xi p+\lambda)+$$
$$+2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+$$
$$+p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]+\lambda\alpha p^3\}+12m\sigma^2_\eta\left\{\{p^3\alpha(\xi p+\lambda)[\lambda p+\right.$$
$$+\xi(\lambda^2+\beta^2)][p^2+4\lambda\xi p+(\lambda^2+\beta^2)]-\xi^2 p^4\alpha(\lambda^2+\beta^2)-\xi p^5\lambda\alpha(\lambda^2+$$
$$+\beta^2)-\lambda^2 p^6\alpha-p^4\alpha(\xi p+\lambda)^2(\lambda^2+\beta^2)\}+p^4\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+\beta^2)](\xi p+\lambda)+ \tag{16}$$
$$+2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+$$
$$\left.+p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]\}\right\}+4p^3m(\xi p+\lambda)[\lambda p+\xi(\lambda^2+\beta^2)][p^2+$$
$$+4\lambda\xi p+(\lambda^2+\beta^2)]-4\xi^2 p^4 m(\lambda^2+\beta^2)-4\xi p^5 m\lambda(\lambda^2+\beta^2)-4\lambda^2 mp^6-$$
$$-4p^4 m(\xi p+\lambda)^2(\lambda^2+\beta^2).$$

Using the notation

$$l=108m\lambda\xi\alpha^3 p^7 \tag{17}$$

$$n=36p^4\alpha m\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+\beta^2)](\xi p+\lambda)+2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]+\lambda\alpha p^3\} \tag{18}$$

$$r=12m\left\{\{p^3\alpha(\xi p+\lambda)[\lambda p+\xi(\lambda^2+\beta^2)][p^2+4\lambda\xi p+(\lambda^2+\beta^2)]-\xi^2 p^4\alpha(\lambda^2+\beta^2)-\xi p^5\lambda\alpha(\lambda^2+\beta^2)-\lambda^2 p^6\alpha-p^4\alpha(\xi p+\lambda)^2(\lambda^2+\beta^2)\}+p^4\{\lambda\alpha[p^2+4\lambda\xi p+(\lambda^2+\beta^2)](\xi p+\lambda)+2p(\xi p+\lambda)^2[p^2+4\lambda\xi p+(\lambda^2+\beta^2)][\lambda p+\xi(\lambda^2+\beta^2)]+\alpha[2p^2\lambda^2+p\xi\lambda(\lambda^2+\beta^2)+(\xi p+\lambda)^2(\lambda^2+\beta^2)]\}\right\} \tag{19}$$

$$s=4p^3m\left\{(\xi p+\lambda)[\lambda p+\xi(\lambda^2+\beta^2)][p^2+4\lambda\xi p+(\lambda^2+\beta^2)]-4\xi^2 p^4(\lambda^2+\beta^2)-4\xi p^5\lambda(\lambda^2+\beta^2)-4\lambda^2 p^6-4p^4(\xi p+\lambda)^2(\lambda^2+\beta^2)\right\}-D\lambda[12\alpha p^2(\xi p+\lambda)-6\alpha\lambda p^2] \tag{20}$$

$$q=-D\lambda\{2p^2(\xi p+\lambda)-8\lambda\xi p(\xi p+\lambda)-2(\xi p+\lambda)(\lambda^2+\beta^2)+2\lambda p^2+2\xi p(\lambda^2+\beta^2)\} \tag{21}$$

obtain the equation

$$l\sigma^8_\eta+n\sigma^6_\eta+r\sigma^4_\eta+s\sigma^2_\eta+q=0\,. \tag{22}$$

We can always find a way to decompose the nonlinear restoring force to one linear component plus a nonlinear component

$$h(\eta)=p^2(\eta+G(\eta)\alpha)\,, \tag{23}$$

where α is the nonlinear factor to control the type and degree of nonlinearity in the system. The idea of linearization is replacing the equation by a linear system:

$$\ddot{\eta}(t)+2\xi_e p_e\dot{\eta}(t)+p_e^2\eta(t)=f(t), \tag{24}$$

where

$$\xi_e=\frac{p}{p_e}\xi. \tag{25}$$

is the damping ratio of equivalent linearized system and p_e is the natural frequency of the equivalent linearized system.

To find an expression for p_e, it is necessary to minimize the expected value of the difference between equations (2) and (24) in a least square sense. Now the difference is the difference between the nonlinear stiffness and linear stiffness terms , which is

$$e=h(\eta(t))-p^2_e\eta(t)\,. \tag{26}$$

The value of p_e can be obtained by minimizing the expectation , of the square error:

$$\frac{dE\{e^2\}}{dp_e^2}=0\,. \tag{27}$$

Substituting the equation (26) into equation (27) performing the necessary differentiation, the expression of p_e can be obtained as:

$$p_e^2=p^2(1+\alpha\frac{E\{\eta G(\eta)\}}{\sigma_\eta^2})=p^2(1+3\alpha\sigma^2_\eta).\,, \tag{28}$$

where σ_η is the standard deviation of $\eta(t)$. This equation shows how the nonlinear component of the stiffness element affects the value of p_e.

3. NUMERICAL RESULTS:

Consider in this example,

$$m = 1kg, k = 36\frac{N}{m}, c = 4\frac{Ns}{m}, \alpha = 3m^{-2}. \tag{29}$$

Let us set the subsequent values of excitation parameters

$$D = 50N^2, \lambda = 1s^{-1}, \beta = 3s^{-1}. \tag{30}$$

Obtain:

$$\sigma^2_\eta = \frac{214 + 1620\sigma^2_\eta}{2692\cdot 10^3\sigma^6_\eta + 7481{,}91\cdot 10^4\sigma^4_\eta + 8354{,}52\cdot 10^3\sigma^2_\eta + 3240}, \tag{31}$$

or

$$2692\cdot 10^3\sigma^8_\eta + 7481{,}91\cdot 10^4\sigma^6_\eta + 8354{,}52\cdot 10^3\sigma^4_\eta + 1620\sigma^2_\eta - 214 = 0, \tag{32}$$

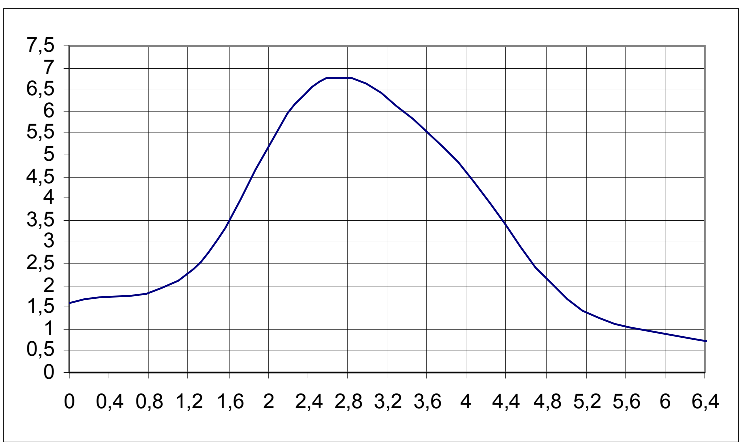
$$\sigma^2_\eta = 0{,}052m^2. \tag{33}$$

Substituting the equation (33) into equation (28), obtain

$$p_e^2 = p^2(1 + 3\alpha\sigma^2_\eta) = 7{,}26s^{-1}. \tag{34}$$

In literature, very little attention has been paid to the frequency domain characteristics of nonlinear, dynamic systems excited by stochastic processes. It will be shown that this information can be of great value for the understanding of the system's stochastic behaviour.

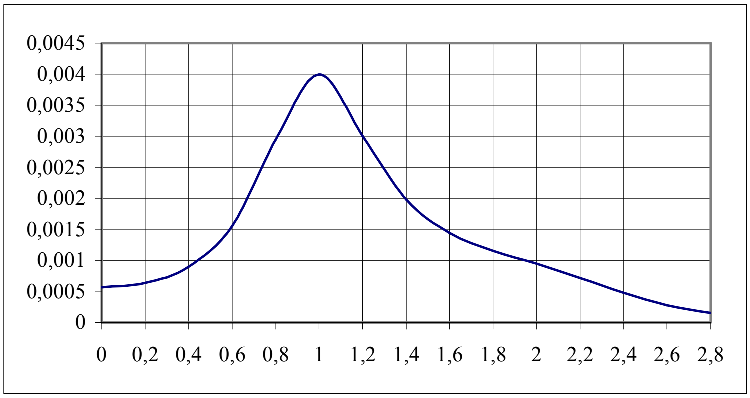
In the figures 1, 2, 3, 4 and 5, the power spectral density of the excitation, $S_F\ [N^2\cdot s]$, is plotted for the differents parameters D, λ, β. Figure 6 describes the harmonic peak with the same parameter values.



$$\frac{\omega}{2\pi}$$

Fig. 5 The power spectral density $S_F\ [N^2\cdot s]$ of excitation

$D = 50N^2, \lambda = 1s^{-1}, \beta = 3s^{-1},\ m = 1kg, k = 36\frac{N}{m}, c = 4\frac{Ns}{m}, \alpha = 3m^{-2}.$



$$\frac{\omega}{2\pi}$$

Fig.6. **The power spectral density** $S_\eta\ [m^2 \cdot s]$ **of response.** $m = 1kg, k = 36\frac{N}{m}, c = 4\frac{Ns}{m}, \alpha = 3m^{-2}$.

3. CONCLUSION

The statistical linearization technique can also tackle a wide variety of problems and also provides approximate information on the frequency domain characteristics of the stochastic response. In this technique, a linear model, which optimally ts the original, nonlinear system (in some statistical sense), is constructed. Due to the fact that response statistics of such a model can, in general, be evaluated analytically, statistical linearization is computationally very effcient. However, it only provides accurate approximation of the response statistics for weakly nonlinear systems. In this chapter, it is shown that the statistical linearization technique structurally underestimates the variance of the response of the piece-wise linear system (even for a moderate nonlinearity). This is dangerous when these estimates are used in failure criteria for practical systems. The cause for this underestimation of the variance can be found by comparing accurate, simulated frequency domain characteristics with those determined using the linear model.

4. REFERENCES:

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