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SOME CONSIDERATION CONCERNING THE USE OF THE LAW OF MIXTURE IN THE IDENTIFICATION OF THE MECHANICAL PROPERTIES OF THE COMPOSITES

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Abstract: *In the case of the composite materials, made by two or more materials, currently is used the law of mixtures to compute the mechanical constants. The relations for these constants, obtained at the beginning of the XIX century by Reuss and Voigt, has the great advantage to write these in terms of percent of the phases, without taking into account the geometrical and structural properties. It is natural for these relations to approximate very good the values of the elastic constants only for some particular cases. The problem is how good are the presented formulas when is necessary to compute the values of the engineering constants and if it is necessary to use other formulas, that have the disadvantage to be complicated. The paper refers to the composite materials reinforced with long fibres in an elastic matrix.*
Key words: *composite materials, upper bounds, lower bounds ,law of mixture.*

1. INTRODUCTION

To determine the elastic constants of a composite material is an important step in the design of a mechanical structure. In literature are presented many calculus methods used for different type of composites. In the paper are presented some method for the calculus of the elastic properties of the composite materials made by two phases and is performed a comparison between these formulae with the laws of mixtures (the laws of mixtures are often used in design due its simplicity). Usually is not possible to obtain analytical expression for the elastic constants and in these cases are used the upper and lower margins for the constants. The problem was studied by a great number of scientists [3]-[8], [11]-[12].

2. ELASTIC MODULI IN TERMS OF PHASES CONCENTRATION (Hill)

This method is one of the most used calculus method due to the fact that require minimum information concerning the two phases. It is necessary to know only the concentration of the phases and the properties of the two material of the composite. It is considered a two phases composite: a matrix with great elasticity and the fibre with very good resistance properties that play a reinforcement role. The fibres are considered very long cylinders oriented along the Ox_1. In the following are presented the properties of the resulting composite considering only the properties of the two constituents and the ratio of the phases. The shape and the dimensions of the reinforcement fibres are not considered. It is considered that are valuable the hypothesis of linear elasticity. The basic idea is to determine the strain energy in two simplified cases, one to obtain an upper limit for the energy and other to obtain a lower limit for the strain energy. Considering the deformation energy of such system, Hill [5]-[7], using the classic models of the elasticity obtains the following bounds for the elastic constants (for a composite reinforced by cylindrical infinite fibres):

$$\frac{\hat{v}_f k_f (k_m + m_m) + \hat{v}_m k_m (k_f + m_m)}{\hat{v}_f (k_m + m_m) + \hat{v}_m (k_f + m_m)} \le k \le \frac{\hat{v}_f k_f (k_m + m_f) + \hat{v}_m k_m (k_f + m_f)}{\hat{v}_f (k_m + m_f) + \hat{v}_m (k_f + m_f)} \qquad (1)$$

$$\hat{v}_f E_f + \hat{v}_m E_m + \frac{4\hat{v}_f \hat{v}_m (\nu_f - \nu_m)^2}{\left(\frac{\hat{v}_f}{k_m} + \frac{\hat{v}_m}{k_f} + \frac{1}{m_m}\right)} \le E \le \hat{v}_f E_f + \hat{v}_m E_m + \frac{4\hat{v}_f \hat{v}_m (\nu_f - \nu_m)^2}{\left(\frac{\hat{v}_f}{k_m} + \frac{\hat{v}_m}{k_f} + \frac{1}{m_f}\right)} \tag{2}$$

$$\hat{v}_f \nu_f + \hat{v}_m \nu_m + \frac{(\nu_f - \nu_m)\hat{v}_f \hat{v}_m \left(\frac{1}{k_m} - \frac{1}{k_f}\right)}{\left(\frac{\hat{v}_f}{k_m} + \frac{\hat{v}_m}{k_f} + \frac{1}{m_2}\right)} \le \nu \le \le \hat{v}_f \nu_f + \hat{v}_m \nu_m + \frac{(\nu_f - \nu_m)\hat{v}_f \hat{v}_m \left(\frac{1}{k_m} - \frac{1}{k_f}\right)}{\left(\frac{\hat{v}_f}{k_m} + \frac{\hat{v}_m}{k_f} + \frac{1}{m_1}\right)} \tag{3}$$

where: $m_f \ge m_m$. It is obvious that in the proposed formulas we try to separate one part that represents the law of mixtures. We mention that these laws are linear in terms of concentration phases.

In the following we present some elastic moduli in three cases. In the first (case nr.1) is considered a composite made by a matrix with the Young modulus 0,4 Mpa and Poisson's ratio equal to 0,35 and the fibre has the Young modulus 10,5 Mpa and the Poisson's ratio 0,22. In fig.1 are presented the bounds of the bulk modulus.

In the second case (nr.2) the composite is considered made by an epoxy resin with the Young modulus 2,7 Mpa and the Poisson's ratio 0,35 and the fibre has the Young modulus 72,4 Mpa and the Poisson's ratio 0,22.

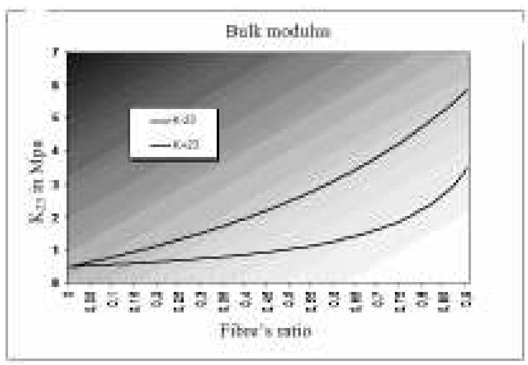

Fig. 1. The upper and lower bounds for the bulk modulus K_{23} in the case nr.1

In the 3rd case are considered the same values for the fibres like in the previous example and the matrix is considered with a Young modulus ten time greater at the same Poisson's ratio. We observe that the properties of the matrix and fibres in the three cases are different but the graphic for the Poisson's ratio is practically the same. The Poisson's ratio is not sensitive relating to the values of the properties of the matrix and fibres.

We can observe that the formulas obtained for the Young modulus calculus are very good and respect, approximatively the law of mixture.

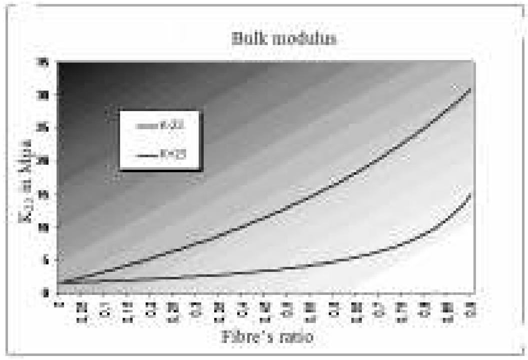

Fig. 2. The upper and lower bounds for the bulk modulus K_{23} in the case nr.2

The relations for the determination of the Poisson's ratio are not so accurate to respect, with a good approximation, the law of mixtures.

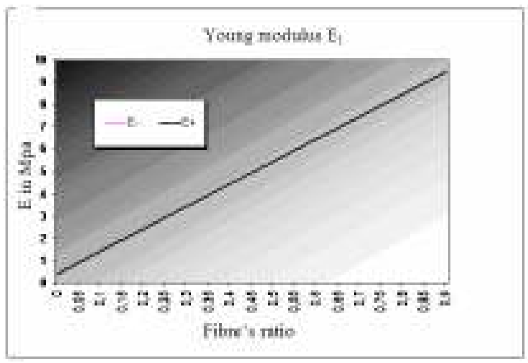

Fig. 3. The upper and lower bound for the Young modulus are practical the same in the case nr.1

3. EXACT SOLUTION FOR SOME PARTICULAR ARRAYS

It is considered that the fibres are parallel disposed. For the calculus is considered a cylindrical specimen long enough where the fibres are parallel with Ox_1 axes. [2]-[3]. The end effects are neglect. Other hypothesis is that the fibres end at the transversal surface considered. In this case is possible to obtain an exact solution for the field of the displacement for a composite cylinder made by a matrix around a cylindrical fibre. If the displacement field is known is possible to determine the bulk modulus, the Young modulus, the transversal modulus and the Poisson's ratio. It is possible to obtain, for bulk modulus, the bounds:

$$K_{23}^{*-} = \frac{K_m}{\dfrac{\hat{v}_1}{m_K} + \hat{v}_2} \leq \quad K_{23}^{*} \leq K_m(m_K \hat{v}_1 + \hat{v}_2) = K_{23}^{*+} \tag{4}$$

$\hat{v}_1$ and $\hat{v}_2$ are the phases fractions:

$$V = V_1 + V_2 \quad ; \qquad \hat{v}_1 + \hat{v}_2 = 1$$
$$\hat{v}_1 = \frac{V_1}{V} \quad ; \quad \hat{v}_2 = \frac{V_2}{V} \quad ;$$

If the fibres are disposed in hexagonal arrangements we have:

$$\hat{v}_1 = \frac{\pi}{2\sqrt{3}} \cong 0,907 \quad ; \quad \hat{v}_2 = 1 - \hat{v}_1 = 0,093$$

and are obtained:

$$K_{23}^{+*} = K_m(0{,}907 m_K + 0{,}093) \quad ;$$

(5)

$$K_{23}^{-*} = K_m \frac{m_K}{0{,}907 + 0{,}093 m_K}$$

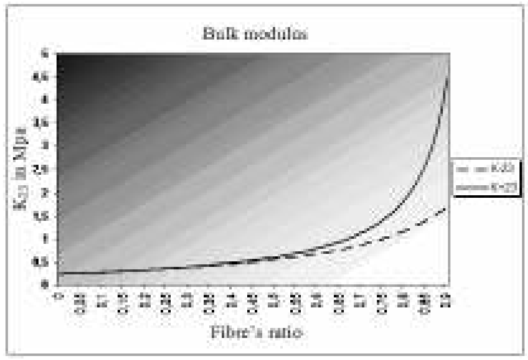

Fig.4. Bulk modulus in the case nr.1 for an hexagonal array

In the first case (nr.1) it is considered a two phases composite made by a matrix with the Young modulus 0.4 Mpa and the Poisson's ratio 0.35 and the fibre has the Young modulus 10.5 Mpa and the Poisson's ratio 0.22. In the figure 4 are represented the upper and lower bounds for the bulk modulus.

In the second case (nr.2) we have considered the composite made by an epoxy resin with the Young modulus 2.7 Mpa and the Poisson's ratio 0.35 and the fibre with the Young modulus 72.4 Mpa and the Poisson's ratio 0.22.

In the 3rd case (nr.3) the fiber is the same that in the precedent case, the matrix has the Young modulus 0.27 Mpa and the Poisson's ratio 0.22.

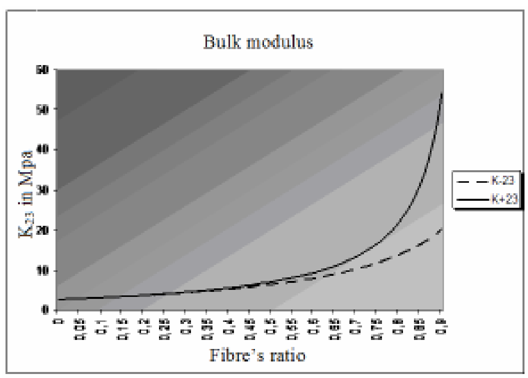

Fig.5. Bulk modulus in the case 2, hexagonal array

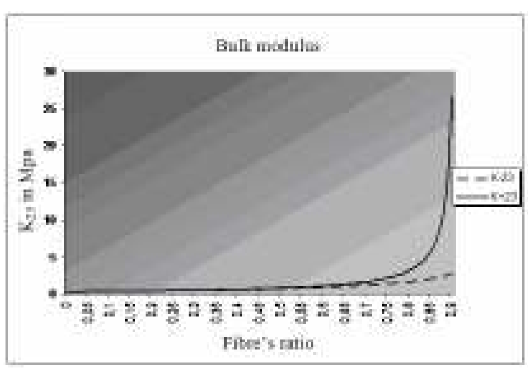

Fig.6. Bulk modulus in the case nr.3, hexagonal array

For the transversal modulus G_{23}^{*} are obtained the relations:

$$G_{23r}^{+} = G_m\left[1 - \frac{2(1-\nu_m)}{1-2\nu_m} v_f A_4^{\varepsilon}\right] \quad ;$$

$$G_{23r}^{-} = G_m\left[1 + \frac{2(1-\nu_m)}{1-2\nu_m} v_f A_4^{\sigma}\right] \quad , \tag{6}$$

and for the Young modulus E_1* :

$$E_{1r}^{*} = m_E E_m \tag{7}$$

This expression is very closed to the law of the mixtures:

$$E_1^{*} = \hat{v}_f E_f + \hat{v}_m E_m \tag{8}$$

and it is easy to apply. If the Poisson's ratio for the fibre and for the matrix is the same, the law of mixture is practically the same with the obtained formulas. In the case of the hexagonal arrays it is difficult to obtain bound for the Poisson's ratio. But is possible to compute this value considering the bounds for the other elastic constants. The relations between these and Poisson's ratio make possible to obtain bound for Poisson's ratio. It is difficult to obtain the results and the formula is not so useful.

4. VARIATIONAL FORMULATIONS

A very used method is the application of the variational calculus to compute the upper and lower bounds for a composite material. Is considered the internal energy for the composite resulting better bounds for the values of the constant elastic for the considered composite [4]. In the following we present some results obtained in [4]:

$$K^{*+} = K_f + \frac{\hat{v}_m}{\dfrac{1}{K_m - K_f} + \dfrac{3\hat{v}_f}{3K_f + 4G_f}} \quad ;$$

$$K^{*-} = K_m + \frac{\hat{v}_f}{\dfrac{1}{K_f - K_m} + \dfrac{3\hat{v}_m}{3K_m + 4G_m}} \tag{9}$$

$$G^{*+} = G_f + \frac{\hat{v}_m}{\dfrac{1}{G_m - G_f} + \dfrac{6(K_f + 2G_f)\hat{v}_f}{5G_f(3K_f + 4G_f)}} \quad ;$$

$$G^{*-} = G_m + \frac{\hat{v}_f}{\dfrac{1}{G_f - G_m} + \dfrac{6(K_m + 2G_m)\hat{v}_m}{5G_m(3K_m + 4G_m)}} \tag{10}$$

These bounds are the best obtained in terms of phase concentrations. We make the observation that if there are great differences between the values of the two phases it results a great difference between the upper and the lower bounds for the computed elastic constants. To obtain better results is necessary that the two phases to have comparable properties or the concentration of one phase to be small.

If the fibres are cylindrical are obtained better results with the relations (the supplementary information concerning the shape of the fibres permit to obtain better results for the upper and lower bounds for the elastic constants):

$$k^{-} = k_m + \frac{\hat{v}_f}{\dfrac{1}{k_f - k_m} + \dfrac{\hat{v}_m}{k_m + G_m}}$$

$$k^{+} = k_f + \frac{\hat{v}_m}{\dfrac{1}{k_m - k_f} + \dfrac{\hat{v}_f}{k_f + G_f}} \qquad (11)$$

$$m^{-} = G_m + \frac{\hat{v}_f}{\dfrac{1}{G_f - G_m} + \dfrac{(k_m + 2G_m)\hat{v}_m}{2G_m(k_m + G_m)}} \quad ;$$

$$m^{+} = G_f + \frac{\hat{v}_m}{\dfrac{1}{G_m - G_f} + \dfrac{(k_f + 2G_f)\hat{v}_f}{2G_f(k_f + G_f)}} \qquad (12)$$

In these cases the obtained relations are semnificatively different from the law of mixtures.

5. CONCLUSIONS

If we make a comparison between the calculus methods we observe that the simplest expression is obtained in the case when only the phase's ratio is considered (for two or more constituents). If supplementary information are considered (for example the cylindricity of the fibres or the ellipticity of fibres) the relations obtained to computes the elastic constants become more complicated. A comparison between the computed values considering these two cases show that in many cases the differences are not significant. In the reality the supplementary information concerning the shape of the fibres are many times very poor due to the extreme variability of the results in the process of the fabrication of fibres. In conclusion the formulas obtained in terms of the concentration ratio can be enough precise to offer the values of elastic constants in many cases. In the paper was presented some of such situation and is indicated when the law of mixture can not offer good approximations.

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