UNIVERSITATEA TRANSILVANIA BRASOV

The 3rd International Conference on International Conference
″Computational Mechanics
and Virtual Engineering″
COMEC 2009
29 – 30 OCTOBER 2009, Brasov, Romania

NEW FORMULATIONS ON SERIAL ROBOTS DYNAMICS

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***Abstract:** In this paper a few important scientific researches of the main author, concerning the matrix exponentials, dynamics matrices, acceleration energy as well as mathematical approach of the polynomial functions referring to trajectory planning for the serial robots will be developed. The acceleration energy and its time derivative can have a great significance in the impulsive (high speed motions) of robots. In order to illustrate the mathematical models, in this paper will be taken into study the mechanical structure of a 2TR serial robot.*
Keywords: *robotics, dynamics, acceleration energy, kinematics, polynomial functions.*

1. THE FORWARD KINEMATIC EQUATIONS

During this section, the main objective consists in finding the equations of forward kinematics that can be applied for the case of any robot structure. The forward kinematics equations are developed based on the matrix exponentials. In the first part of this paper, the forward kinematics equations (linear and angular velocities and accelerations) are written in an exponential form, then the obtained equations are applied to determine the expression for the acceleration energy.

1.1 Matrix Exponential in Forward Kinematics

According to [3], in the following, are presented some important expressions that result from applying the *Algorithm of Matrix Exponentials in Kinematics (MEK)*. The extended matrix of nominal geometry is given as input data. In the first step, for $i=1\rightarrow n$ are determined the screw parameters, $\bar{k}_i^{(0)}, \bar{v}_i^{(0)}$ and the column vector, represented by $\bar{b}_i$. Applying the algorithm of *Matrix Exponentials* in robot geometry, finally is obtained the column vector of the operational coordinates containing both, the position and the orientation for the robot end - effectors. These results are further used in determining the equations of direct kinematics and the *Jacobian* matrix as well, for any robot structure, regardless complexity.

$$^{0}J\left(\bar{\theta}\right)=\left\{\left[{}^{0}J_{iv}\left(\bar{\theta}\right)\right]^{T}\ \left[{}^{0}J_{i\Omega}\left(\bar{\theta}\right)\right]^{T}\ \left\{i=1\rightarrow n\right\}\right\}^{T} \tag{1}$$

The components of the Jacobian matrix can be also written in an explicit form, as it is presented in the expressions bellow:

$$\left\{\begin{matrix}\left[{}^{0}J_{i\Omega}\left(\bar{\theta}\right)\ \left(i=1\rightarrow n\right)\right]= \\ \left(exp\left\{\sum_{j=0}^{i-1}\left\{\bar{k}_j^{(0)}\times\right\}q_j\cdot\Delta_j\right\}\cdot\bar{k}_i^{(0)}\cdot\Delta_i\right)\end{matrix}\right\} \tag{2}$$

$$\left(\begin{matrix}\left\{\begin{matrix}\left[{}^{0}J_{iv}\left(\bar{\theta}\right)\right] \\ \left\{i=1\rightarrow n\right\}\end{matrix}\right\}=exp\left\{\sum_{j=0}^{i-1}\left\{\bar{k}_j^{(0)}\times\right\}q_j\cdot\Delta_j\right\}\cdot\bar{v}_i+exp\left\{\sum_{k=i}^{n}\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\cdot\bar{p}_n^{(0)}+ \\ +\Delta_i\cdot exp\left\{\sum_{j=0}^{i-1}\left\{\bar{k}_j^{(0)}\times\right\}q_j\cdot\Delta_j\right\}\cdot\left\{\bar{k}_i^{(0)}\times\right\}\left\{\sum_{k=i}^{n}\left\{exp\left\{\sum_{m=i-1}^{k-1}\left\{\bar{k}_m^{(0)}\times\right\}q_m\cdot\delta_m\right\}\cdot\bar{b}_k\right\}\right\}\end{matrix}\right) \tag{3}$$

The column vector $\bar{b}_i$, established with the following expression, can be defined in the matrix form such as presented:

$$\bar{b}_i=\left\{I_3\cdot q_i+\left\{\bar{k}_i^{(0)}\times\right\}\left[1-c\left(q_i\cdot\Delta_i\right)\right]+\bar{k}_i^{(0)}\cdot\bar{k}_i^{(0)T}\cdot\left[q_i-s\left(q_i\cdot\Delta_i\right)\right]\right\}\cdot\bar{v}_i^{(0)} \tag{4}$$

In the expressions, presented above, $\bar{k}_j^{(0)}$ is the unit vector, that describes the orientation of each driving axis of the robot mechanical structure; also, q_j represents the generalized coordinate, or the so called geometrical control function of any kinematical driving axis; $\bar{p}_n$ describes the position of the end-effector with respect to $\{0\}$ fixed frame, the parameters $\bar{k}_i^{(0)}$ and $\bar{v}_i^{(0)}$ are the screw parameters, and finally, Δ_i represents a matrix operator whose value depends on the type of motion performed around and along the driving axis.

According to MEK Algorithm, are determined some matrix exponentials, characterized by the following expressions:

$$\underset{(3\times3)}{ME}\left(V_{i1}\right)=\exp\left\{\sum_{j=0}^{i-1}\left\{\bar{k}_j^{(0)}\times\right\}q_j\cdot\Delta_j\right\} \quad (5)$$

$$\underset{(3\times6)}{ME}\left(V_{i2}\right)=\left[\; I_3 \quad \Delta_i\cdot\left\{\bar{k}_i^{(0)}\times\right\}\;\right] \quad (6)$$

$$\underset{\{6\,x\,[9+3\cdot(n-i)]\}}{ME}\left(V_{i3}\right)\equiv\begin{bmatrix} I_3 & [0] \\ [0] & \left[\exp\left\{\sum_{m=i-1}^{k-1}\left\{\bar{k}_m^{(0)}\times\right\}q_m\cdot\delta_m\cdot\Delta_m\right\}\right] \\ [0] & \exp\left\{\sum_{k=i}^{n}\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\} \end{bmatrix} \quad (7)$$

Applying some matrix transformations, into expressions *(5) - (7),* finally, are obtained the following matrix exponential:

$$\underset{(6\times6)}{ME}\left\{J_{i1}\right\}=\begin{bmatrix} ME\{V_{i1}\} & [0] \\ [0] & ME\{V_{i1}\}\end{bmatrix} \quad (8) \qquad \underset{(6\times9)}{ME}\left\{J_{i2}\right\}=\begin{bmatrix} ME\{V_{i2}\} & [0] \\ [0] & I_3\end{bmatrix} \quad (9) \qquad ME\left\{J_{i3}\right\}=\begin{bmatrix} ME\{V_{i3}\} & [0] \\ [0] & I_3\end{bmatrix} \quad (10)$$

The *absolute* values for *angular and linear velocities* and *accelerations*, corresponding to any kinetic link $i=1\rightarrow n$ from the mechanical robot structure, are determined by means of the expressions which are presented in the following:

$${}^0\overline{\omega}_i\equiv\sum_{j=1}^{i}\left\{\exp\left\{\sum_{k=1}^{j-1}\left\{\bar{k}_k^{(0)}\times\right\}q_k\cdot\Delta_k\right\}\right\}\cdot\bar{k}_j^{(0)}\cdot\dot{q}_j\cdot\Delta_j\,; \qquad {}^0\dot{\overline{\omega}}_i\equiv\sum_{j=1}^{i}\left\{ME\left\{V_{j1}\right\}\cdot\ddot{q}_j+ME\left\{\dot{V}_{j1}\right\}\cdot\dot{q}_j\right\}\cdot\bar{k}_j^{(0)}\cdot\Delta_j\,. \quad (11)$$

$${}^0\overline{v}_i\equiv\sum_{j=1}^{i}ME\left\{J_{j1}\right\}\cdot ME\left\{J_{j2}\right\}\cdot ME\left\{J_{j3}\right\}\cdot M_{jv}\cdot\dot{q}_j\,; \qquad {}^0\dot{\overline{v}}_i\equiv\sum_{j=1}^{i}\left\{\underset{(i=1\rightarrow n)}{{}^0\dot{J}_i}\cdot\dot{q}_j\right\}+\sum_{j=1}^{i}\left\{ME\left\{J_{j1}\right\}\cdot ME\left\{J_{j2}\right\}\cdot ME\left\{J_{j3}\right\}\cdot M_{jv}\cdot\ddot{q}_j\right\}. \quad (12)$$

The expressions for the *Jacobian* matrix written in exponential form are substituted in the generalized expression (determined above) in order to obtain the operational velocities and accelerations, described by relations bellow:

$$\left[{}^{(n)0}\dot{X} \quad {}^{(n)0}\ddot{X}\right]^T\equiv\left\{\begin{bmatrix}\left[{}^{(n)0}\overline{v}_n^T \quad {}^{(n)0}\overline{\omega}_n^T\right]^T \\ \left[{}^{(n)0}\dot{\overline{v}}_n^T \quad {}^{(n)0}\dot{\overline{\omega}}_n^T\right]^T\end{bmatrix}\equiv\begin{bmatrix}[0] & {}^{(n)0}J\left(\overline{\theta}\right) \\ {}^{(n)0}J\left(\overline{\theta}\right) & {}^{(n)0}\dot{J}\left(\overline{\theta}\right)\end{bmatrix}\cdot\begin{bmatrix}\ddot{\overline{\theta}} \\ \dot{\overline{\theta}}\end{bmatrix}\right\} \quad (13)$$

Remarks: The matrix exponentials enjoy important advantages due to their compact form, easy geometric visualization and also, they avoid the frames typical to every kinetic link. The matrix exponentials have an essential role in determining of the dynamic control functions that characterize mechanical robot structures, regardless of complexity.

2. THE KINEMATICAL CONTROL FUNCTIONS

The kinematical control functions refer to the time functions for the generalized variables from every driving joint. They describe, in each moment, the robot configuration in keeping with the absolute motion of the end-effector, included in the input data. According to [1], [2], [4] and [5], the kinematical control functions are determined using the differential expressions based on Jacobian matrix or polynomial interpolating functions of various order, which are further presented .

2.1 Control Functions based Jacobian Matrix

According to [2] and [4], in the case of the application of the *DKM* equations based on the Jacobian matrix and its time derivative, the kinematical control functions are determined with the following matrix expression:

$$\left[\dot{\overline{\theta}}^T(t) \quad \ddot{\overline{\theta}}^T(t)\right]^T=Matrix\left\{{}^0J\left[\overline{\theta}(t)\right]^{-1}\right\}\cdot\left\{{}^0\ddot{X}^T(t) \quad {}^0\dot{X}^T(t) \quad \left\{{}^0\dot{J}\left[\overline{\theta}(t)\right]\cdot\dot{\overline{\theta}}\right\}^T\right\} \quad (14)$$

.In the above equation ${}^{(n)0}J\left(\overline{\theta}\right)^{-1}$ represents the inverse of the Jacobian matrix. According to [1], [2] and [4] the kinematical singularities for any robot type occurs when: $Det\left[{}^{(n)0}J\left(\overline{\theta}\right)\right]\equiv0$. By applying some matrix transformations, the inverse of the Jacobian matrix can be determined, according to the following expressions:

$${}^0J\left(\overline{\theta}\right)^{-1}=\left[{}^0J\left(\overline{\theta}\right)^T\cdot{}^0J\left(\overline{\theta}\right)\right]^{-1}\cdot{}^0J\left(\overline{\theta}\right)^T\,; \qquad {}^0J\left(\overline{\theta}\right)^{-1}={}^0J\left(\overline{\theta}\right)^T\cdot\left[{}^0J\left(\overline{\theta}\right)\cdot{}^0J\left(\overline{\theta}\right)^T\right]^{-1} \quad (15)$$

Remarks: In the expressions (15), for the same robot configuration there are imposed the following mathematical conditions:

$$Det\left\{\left[{}^0J\left(\overline{\theta}\right)^T\cdot{}^0J\left(\overline{\theta}\right)\right]^{-1}\right\}=\{0\;;Det\neq0\}\,; \qquad Det\left\{\left[{}^0J\left(\overline{\theta}\right)\cdot{}^0J\left(\overline{\theta}\right)^T\right]^{-1}\right\}=\{0\;;Det\neq0\} \quad (16)$$

If the above conditions are not satisfied, [3], [4], the control functions are defined based on the polynomial interpolating functions.

2.2 The Polynomial Interpolating Functions

In keeping with [2] and [4], this section is devoted to the presenting of the algorithm for the kinematical control functions based the polynomial interpolating functions of $(3 \cdot m)$ type, with restrictions.

2.2.1 Polynomial Functions of $(3 \cdot m)$ restrictions

Within this section are presented a few important expressions necessary to determine the kinematical control functions based on the interpolating polynomial with $(3 \cdot m)$ kinematical restrictions. They are characterized by means of the spline cubic functions for the generalized coordinates from the driving joints. In the view of this for the generalized accelerations are imposed polynomial interpolating function of first order. This algorithm is described according to [1], [2] and [4]. The matrix of the unknown coefficients is determined on the basis of the following expression:

$$\underset{[(m-1)\times(m-1)]}{Matrix\ \{A\}} = \left\{ A_1 \ \left[A_k \quad k = 3 \to m-3 \right]^T \ A_{m-1} \right\}^T \tag{17}$$

In the relation above, the sub matrix A_1, having $[2 \times (m-1)]$ size can be characterized by means of the following expression:

$$A_1 = \left[\begin{array}{ccc|c} 3 \cdot t_1 + 2 \cdot t_2 + \dfrac{t_1^2}{2} & t_2 & 0 & \\ & & & [0]^{m-4} \\ t_2 - \dfrac{t_1^2}{t_2} & 2 \cdot (t_2 + t_3) & t_3 & \end{array} \right]; \qquad A_k = \left[[0]^{k-2} \ t_k \ \ 2 \cdot (t_k + t_{k+1}) \ t_{k+1} \ [0]^{m-(k+2)} \right] \tag{18}$$

In the above expression, $[0]^{m-4}$ represents a $[2 \times (m-4)]$ zero matrix. The row matrix A_k, with $[1 \times (m-1)]$ size contains the components presented in (18), as time functions, where $[0]^{k-2}$ is a zero matrix having: $1 \times (k-2)$ size. The sub matrix A_{m-1}, having $[2 \times (m-1)]$ size is characterized by means of the following expressions:

$$A_{m-1} = \left[[0]^{m-4} \quad A_{m-1}(t) \right]; \qquad \text{where:} \quad A_{m-1}(t) = \begin{bmatrix} t_{m-2} & 2 \cdot (t_{m-2} + t_{m-1}) & t_{m-1} - \dfrac{t_m^2}{t_{m-1}} \\ 0 & t_{m-1} & 3 \cdot t_m + 2 t_{m-1} + \dfrac{t_m^2}{t_{m-1}} \end{bmatrix} \tag{19}$$

The column vector of the unknowns described by X_j and free terms B_j respectively are determined with the expressions:

$$X_j = \left[\ddot{q}_{jk} \ ; \ k = 1 \to n-1 \right]^T; \qquad B_j = \left[b_{j1} \ b_{j2} \ \left[b_{jk} \ ; \ k = 3 \to n-3 \right] \ b_{jm-2} \ b_{jm-1} \right]^T \tag{20}$$

where: $b_{j1} = 6 \cdot \left[\dfrac{q_{j2} - q_{j0}}{t_2} - \dot{q}_{j0} \cdot \dfrac{t_1 + t_2}{t_2} - \ddot{q}_{j0} \cdot \left(\dfrac{t_1}{2} + \dfrac{1}{3} \cdot \dfrac{t_1^2}{t_2} \right) \right]$; $\quad b_{j2} = 6 \cdot \left(\dfrac{q_{j0} - q_{j2}}{t_2} - \dfrac{q_{j3} - q_{j2}}{t_3} + \dot{q}_{j0} \cdot \dfrac{t_1}{t_2} + \dfrac{1}{3} \dfrac{t_1^2}{t_2} \cdot \ddot{q}_{j0} \right)$;

$$b_{jk} = 6 \cdot \left(\frac{q_{jk+1} - q_{jk}}{t_{k+1}} - \frac{q_{jk} - q_{jk-1}}{t_k} \right); \qquad b_{jm-2} = 6 \cdot \left(\frac{q_{jm-3} - q_{jm-2}}{t_{m-2}} + \frac{q_{jm} - q_{jm-2}}{t_{m-1}} - \dot{q}_{jm} \cdot \frac{t_m}{t_{m-1}} + \frac{1}{3} \cdot \frac{t_m^2}{t_{m-1}} \cdot \ddot{q}_{jm} \right);$$

$$b_{jm-1} = 6 \cdot \left[\frac{q_{jm-2} - q_{jm}}{t_{m-1}} + \dot{q}_{jm} \cdot \frac{t_{m-1} + t_m}{t_{m-1}} - \ddot{q}_{jm} \left(\frac{t_m}{2} + \frac{1}{3} \cdot \frac{t_m^2}{t_{m-1}} \right) \right]$$

The unknowns are represented by the generalized accelerations X_j, which are obtained based on the equation bellow:

$$\left\{ X_j \equiv \left[\ddot{q}_{jk} \ ; \ k = 1 \to n-1 \right]^T \right\} = A^{-1} \cdot B_j ; \tag{21}$$

where A^{-1} represents the inverse of the matrix A. This is because the condition: $[t_i > 0 \ \ i = 1 \to n]$ is satisfied.

For $k = 1 \to m-1$, considering the results (24), the generalized velocities can be calculated with the following equation:

$$\left\{ \dot{q}_{jk} = \frac{1}{3} \cdot \ddot{q}_{jk} \cdot t_k + \frac{1}{6} \cdot \ddot{q}_{jk-1} \cdot t_k + \frac{1}{t_k} \cdot \left(q_{jk} - q_{jk-1} \right) \right\}; \ \ddot{q}_{jk} \in X_j \ . \tag{22}$$

Considering (21) and (22), the kinematical control functions are determined, in keeping with the expression bellow:

$$\left\{ q_{jk}(\tau) = \frac{(\tau_k - \tau)^3}{6 \cdot t_k} \cdot \ddot{q}_{jk-1} + \frac{(\tau - \tau_{k-1})^3}{6 \cdot t_k} \cdot \ddot{q}_{jk} + \left(\frac{q_{jk}}{t_k} - \frac{t_k}{6} \cdot \ddot{q}_{jk} \right) \cdot (\tau - \tau_{k-1}) + \left(\frac{q_{jk-1}}{t_k} - \frac{t_k}{6} \cdot \ddot{q}_{jk-1} \right) \cdot (\tau_k - \tau) \right\}; \tag{23}$$

$$\left\{ \dot{q}_{jk}(\tau) = -\frac{(\tau_k - \tau)^2}{2 \cdot t_k} \cdot \ddot{q}_{jk-1} + \frac{(\tau - \tau_{k-1})^2}{2 \cdot t_k} \cdot \ddot{q}_{jk} - \left(\frac{q_{jk-1}}{t_k} - \frac{t_k}{6} \cdot \ddot{q}_{jk-1} \right) + \left(\frac{q_{jk}}{t_k} - \frac{t_k}{6} \cdot \ddot{q}_{jk} \right) \right\};$$

$$\left\{ \ddot{q}_{jk}(\tau) = \frac{\tau_k - \tau}{t_k} \cdot \ddot{q}_{jk}(\tau_{k-1}) + \frac{\tau - \tau_{k-1}}{t_k} \cdot \ddot{q}_{jk}(\tau_k) \right\}. \tag{24}$$

Remark: In conclusion, using interpolating polynomial functions of $(3 \cdot m)$ type, the kinematical control functions are found.

3. THE DYNAMIC EQUATIONS

The fundamental theorems from complex systems dynamics, where robotic structures are also included, play an essential role in determining the matrix equations of dynamics and of dynamic control functions (generalized driving forces). These theorems are based the fundamental notions from the mechanical system dynamics, such as: the mechanical impulse, mechanical work, and angular momentum, kinetic energy and acceleration energy, according to [4].

3.1 The Generalized Active Forces

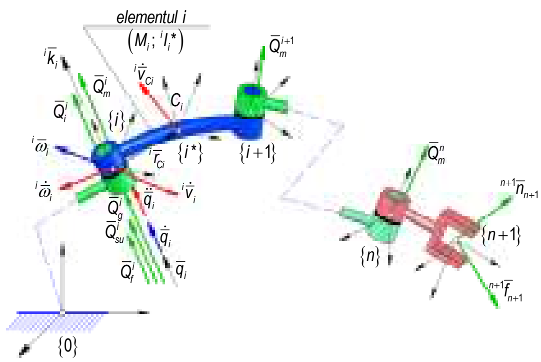

Figure 1: The generalized dynamic forces

The active forces are usually a result of gravitational load applied in the mass center of robot. According to [4], the *generalized gravitational force* is defined as:

$$Q_g(\bar{\theta}) = \left\{ \begin{array}{c} \left[Q_g^i = {}^{(n)0}J_i^T \cdot {}^{(n)0}\nearrow_{X_i};\ i = 1 \to n \right]^T \equiv \\ \equiv {}^{(n)0}J(\bar{\theta})^T \cdot {}^{(n)0}\nearrow_{X_i}(\bar{\theta}) \end{array} \right\} \tag{25}$$

$${}^{(n)0}\nearrow_{X_i} = \left\{ \begin{array}{c} {}^{(n)0}\bar{F}_{X_i} \\ \hline {}^{(n)0}\bar{N}_{X_i} \end{array} \right\} = \left\{ \begin{array}{c} \sum\limits_{j=i}^{n} M_j \cdot {}^{(0)n}_{\ \ n}[R]^T \cdot \bar{g} \\ \hline \sum\limits_{j=i}^{n} M_j \cdot {}^{(0)n}_{\ \ n}[R]^T \cdot \left[\left({}^0\bar{r}_{C_j} - \bar{p}_n \right) \times \bar{g} \right] \end{array} \right\} \tag{26}$$

The *generalized manipulating* force applied in the last kinematical chain of the robot introduces a generalized force in each kinetic axis of the robot, affecting the static and dynamic behavior of robot structures:

$$Q_{SU}(\bar{\theta}) \equiv \left\{ {}^{(n)0}J(\bar{\theta}) \cdot {}^{(n)0}\nearrow_{X_i} \right\} = \left[Q_{SU}^i = {}^{(n)0}J_i^T \cdot {}^{(n)0}\nearrow_{X_i},\ \ i = 1 \to n \right]^T \quad \text{where: } Q_{SU}^i \equiv {}^nJ_i^T \cdot {}^n\nearrow_{X_i} \tag{27}$$

The resultant force-momentum vector is applied with respect to $\{n\}$ frame. This is given by the following expression:

$$\left\{ {}^{(n)0}\nearrow_{X_i} = \begin{bmatrix} {}^{(n)0}\bar{F}_X \\ ----- \\ {}^{(n)0}\bar{N}_X \end{bmatrix} \equiv \begin{bmatrix} {}^{(n)0}_{n+1}[R] & [0] \\ \left\{ {}^{(0)n}_{\ \ n}[R]^T \cdot \bar{p}_{n+1n} \times \right\} \cdot {}^{(n)0}_{n+1}[R] & {}^{n}_{n+1}[R] \end{bmatrix} \cdot \begin{bmatrix} {}^{n+1}\bar{f}_{n+1} \\ {}^{n+1}\bar{n}_{n+1} \end{bmatrix} \right\} \tag{28}$$

Generalized driving forces from each driving axis are the unknowns of inverse dynamic modeling used in robot control. According to [2] and [4], the iterative algorithm of the generalized driving forces is applied. In the first step, using external iterations, are determined the inertia forces and torque characteristic to any kinetic assembly. In the second step, for $i = n \to 1$, are obtained (internal iterations), the resultant connecting forces and their moments, from each driving joint.

$${}^i\bar{f}_i = \Delta_m^2 \left\{ \Delta_\theta \sum_{j=i}^{n} {}^0_i[R]^T \cdot {}^j\bar{F}_j + \sum_{j=i}^{n} M_j \cdot {}^0_i[R]^T \cdot \bar{g} \right\} + (-1)^{\Delta M} \cdot \frac{1 - \Delta_m}{1 + \Delta_m} \cdot {}^{i}_{n+1}[R] \cdot {}^{n+1}\bar{f}_{n+1} \tag{29}$$

$${}^i\bar{n}_i = \left\{ \begin{array}{c} \Delta_m^2 \cdot \left\{ \Delta_\theta \left\{ \sum\limits_{j=i}^{n} {}^0_i[R]^T \cdot \left({}^0\bar{r}_{C_j} - \bar{p}_i \right) \times {}^i_j[R] \cdot {}^j\bar{F}_j + {}^i_j[R] \cdot {}^j\bar{N}_j \right\} + \sum\limits_{j=i}^{n} M_j \cdot {}^0_i[R]^T \cdot \left[\left({}^0\bar{r}_{C_j} - \bar{p}_i \right) \times \bar{g} \right] \right\} + \\ + (-1)^{\Delta_m} \cdot \dfrac{1 - \Delta_m}{1 + \Delta_m} \cdot \left\{ {}^0_i[R]^T \cdot (\bar{p} - \bar{p}_i) \times {}^{i}_{n+1}[R] \cdot {}^{n+1}\bar{f}_{n+1} + {}^{i}_{n+1}[R] \cdot {}^{n+1}\bar{n}_{n+1} \right\} \end{array} \right\} \tag{30}$$

The expression of the *generalized driving force*, from each driving axis, can be written in the following matrix form:

$$Q_m^i = \left\{ {}^i\bar{f}_i^T (1 - \Delta_i) + {}^i\bar{n}_i^T \Delta_i \right\} \cdot {}^i\bar{k}_i \equiv \Delta_m^2 \left\{ \Delta_\theta \cdot Q_{i\nearrow}^i + Q_g^i \right\} + (-1)^{\Delta m} \cdot \frac{1 - \Delta_m}{1 + 3 \cdot \Delta_m} \cdot Q_{su}^i \tag{31}$$

The *generalized inertia forces* that characterize each driving joint from the robot structure can be written as follows:

$$Q_{i\nearrow}(\bar{\theta})=\begin{bmatrix} Q^i_{i\nearrow}={}^0J^T_i\cdot{}^0\overline{\nearrow}^*_{X_i} \\ i=1\rightarrow n \end{bmatrix}^T ; \quad {}^{(n)0}\overline{\nearrow}^*_{X_i}=\begin{Bmatrix} {}^{(n)0}\bar{F}^*_{X_i} \\ \text{-----} \\ {}^{(n)0}\bar{N}^*_{X_i} \end{Bmatrix}=\begin{Bmatrix} \sum_{j=i}^{n}{}^{(n)0}_{\;\;j}[R]\cdot{}^j\bar{F}_j \\ \text{--------------------} \\ \sum_{j=i}^{n}\left[{}^{(0)n}_{\;\;n}[R]^T\left({}^0\bar{r}_{C_j}-\bar{p}_i\right)\times{}^{(n)0}_{\;\;j}[R]\cdot{}^j\bar{F}_j+{}^{(n)0}_{\;\;j}[R]\cdot{}^j\bar{N}_j\right] \end{Bmatrix} \quad (32)$$

where, $${}^j\bar{F}_j=M_j\cdot\left\{{}^j\dot{\bar{v}}_j+{}^j\dot{\bar{\omega}}_j\times{}^j\bar{r}_{C_j}+{}^j\bar{\omega}_j\times{}^j\bar{\omega}_j\times{}^j\bar{r}_{C_j}\right\}; \qquad {}^j\bar{N}_j=\left\{{}^jI^*_j\cdot{}^j\dot{\bar{\omega}}_j+{}^j\bar{\omega}_j\times{}^jI^*_j\cdot{}^j\bar{\omega}_j\right\} \quad (33)$$

The resultant vector of the generalized inertia forces and moments, in the interval $[i\rightarrow n]$ with respect to the origin of the frame $\{n\}$, in the Cartesian space is determined by applying the expression (32) whose components are given by (33).

4. THE ACCELERATION ENERGY IN THE ROBOT DYNAMICS

The dynamics equations, according to [2] and [4], in case of the robots with $(n\ d.o.f)$, subjected to holonomic or non-holonomic relation, can be determined on the basis of a fundamental dynamics quantity, the *acceleration energy*:

$$\begin{Bmatrix} E^j_A(q_k;\dot{q}_k;\ddot{q}_k;\ k=1\rightarrow j)\equiv(-1)^{\Delta_M}\dfrac{1-\Delta_M}{1+3\cdot\Delta_M}\cdot\left\{\dfrac{1}{2}\cdot M_j\cdot{}^j\dot{\bar{v}}^T_{C_j}\cdot{}^j\dot{\bar{v}}_{C_j}\right\}+\Delta^2_M\left\{\dfrac{1}{2}\cdot{}^j\dot{\bar{\omega}}^T_j\cdot\left\{{}^jI^*_j\cdot{}^j\dot{\bar{\omega}}_j+\left[{}^j\bar{\omega}_j\times{}^jI^*_j\cdot{}^j\bar{\omega}_j\right]\right\}+\right. \\ \left.\dfrac{1}{2}\cdot{}^j\dot{\bar{\omega}}^T_j\cdot\left[{}^j\bar{\omega}_j\times{}^jI^*_j\cdot{}^j\bar{\omega}_j\right]+\dfrac{1}{2}\cdot\bar{\omega}^T_j\cdot\left[{}^j\bar{\omega}^T_j\cdot Tr.\left({}^jI_{pj}\right){}^j\bar{\omega}_j-{}^j\bar{\omega}^T_j\cdot{}^jI_{pj}\cdot{}^j\bar{\omega}_j\right]{}^j\bar{\omega}_j\right\} \end{Bmatrix} \quad (34)$$

The final expression, obtained above, represents *the generalized and explicit form of acceleration energy*. In the explicit form, the following operator is applied: $\Delta_M=\{\{-1;\ General\ \ Motion\};\{0;Translation\ \ Motion\};\{1;Rotation\ \ Motion\}\}$.

In keeping with [4], extending the study about the mechanical system with n d.o.f., and applying differential transformations, it results a new expression for the acceleration energy in matrix form. This expression is determined based on dynamics matrices, having the following significance: $M(\bar{\theta};\dot{\bar{\theta}})$ is the mass matrix, $V(\bar{\theta};\dot{\bar{\theta}})$ the column *vector* containing the *Coriolis* and *centrifugal terms*, $D(\bar{\theta};\dot{\bar{\theta}})$, represents the *pseudoinertial matrix* of the acceleration energy.

$$E_A(\bar{\theta};\dot{\bar{\theta}};\ddot{\bar{\theta}})=\frac{1}{2}\cdot\left[\ddot{\bar{\theta}}^T\cdot M(\bar{\theta})\cdot\ddot{\bar{\theta}}+\ddot{\bar{\theta}}^T\cdot V(\bar{\theta};\dot{\bar{\theta}})+\ddot{\bar{\theta}}^T\cdot D(\bar{\theta};\dot{\bar{\theta}})\cdot\bar{\theta}\right] \quad (35)$$

This equation contains all the *MD-type* properties, as well as the acceleration of the mass center, angular rotation velocity and acceleration.

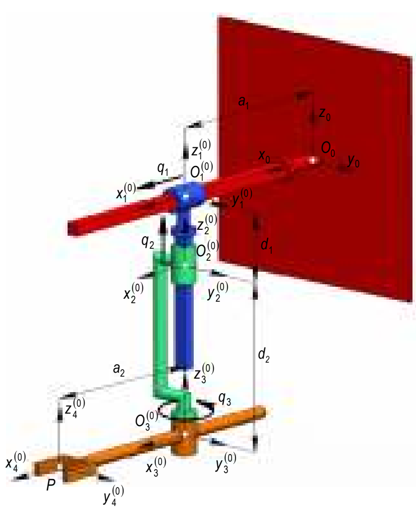

Figure 2 The kinematic diagram of a 2TR robot

4.1 Application of the Acceleration Energy on 2TR-type Structure

The Algorithm of Matrix Exponentials was applied on a 2TR robot structure (Fig.2) resulting the geometry and kinematical equations. The analysis of the 2TR structure was extended in dynamics, obtaining the expression for the acceleration energy, written in a generalized and explicit form. Performing some differential transformations on the expression of acceleration energy, are obtained the equations of motion for the 2TR robot structure. In the first time the expression of Jacobian matrix, relative to $\{0\}$ and the resultant Jacobian matrix projected on $\{n\}$ moving system for 2TR robot is established as:

$${}^0J(\bar{\theta})=\begin{bmatrix}1&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{bmatrix}^T;\quad {}^nJ(\bar{\theta})^T=\begin{bmatrix}-cq_3&-sq_3&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&1\end{bmatrix} \quad (36)$$

The angular and linear velocities and accelerations, relative to frame $\{i\}$, are:

$${}^3\bar{\omega}_3\equiv[0\quad 0\quad \dot{q}_3]^T;\qquad {}^3\dot{\bar{\omega}}_3\equiv[0\quad 0\quad \ddot{q}_3]^T;\qquad {}^3\bar{v}_3\equiv[\dot{q}_1\quad 0\quad \dot{q}_2]\qquad {}^3\dot{\bar{v}}_3\equiv[\ddot{q}_1\quad 0\quad \ddot{q}_2]$$

The angular and linear velocities and accelerations projected on $\{3\}$ moving frame, for the *2TR* mechanical structure are equal with the angular and linear velocities with respect to $\{0\}$ frame. The operational velocities and accelerations are:

$$\left\{{}^{(3)0}\dot{\bar{X}}=\left[{}^{(3)0}\bar{v}^T_3\quad {}^{(3)0}\bar{\omega}^T_3\right]^T=[\dot{q}_1\quad 0\quad \dot{q}_2\quad 0\quad 0\quad \dot{q}_3]^T;\quad {}^{(3)0}\ddot{\bar{X}}=\left[{}^{(3)0}\dot{\bar{v}}^T_3\quad {}^{(3)0}\dot{\bar{\omega}}^T_3\right]^T=[\ddot{q}_1\quad 0\quad \ddot{q}_2\quad 0\quad 0\quad \ddot{q}_3]^T\right\} \quad (37)$$

where, $\left({}^{(3)0}\bar{v}_3^T,\ {}^{(3)0}\dot{\bar{v}}_3^T\right)$ are linear velocities and accelerations of end-effector relative to $\{0\}$ and $\{3\}$ frames, and $\left({}^{(3)0}\bar{\omega}_3^T,\ {}^{(3)0}\dot{\bar{\omega}}_3^T\right)$ are the angular velocities and acceleration. The expression of the acceleration energy for the 2TR robot:

$$E_A\left[\bar{\theta}(t);\ \dot{\bar{\theta}}(t);\ \ddot{\bar{\theta}}(t)\right] \equiv \left\{\begin{array}{c} \frac{1}{2}\cdot(M_1+M_2+M_3)\cdot\ddot{q}_1^2+\frac{1}{2}\cdot(M_2+M_3)\cdot\ddot{q}_2^2+\frac{1}{2}\cdot M_3\cdot a_2^2\cdot\ddot{q}_3^2-\left(M_3\cdot a_2\cdot\dot{q}_3^2\cdot cq_3\right)\cdot\ddot{q}_1- \\ -\left(M_3\cdot a_2\cdot sq_3\right)\cdot\ddot{q}_1\cdot\ddot{q}_3+\frac{1}{2}\cdot M_3\cdot a_2^2\cdot\dot{q}_3^4+\frac{1}{2}\cdot I_z\cdot\ddot{q}_3^2+\frac{1}{2}\cdot\left({}^3I_{xx}+{}^3I_{yy}\right)\cdot\dot{q}_3^4 \end{array}\right\} \quad (38)$$

where M_1, M_2 and M_3 are the mass of the kinetic elements, I_z is the axial inertia moment on *z* axis and ${}^3I_{xx}, {}^3I_{yy}$ are planar inertia mechanical moments. The dynamic equations for the 2TR type robot, based on acceleration energy , according to [4] are:

$$\frac{\partial E_A}{\partial\ddot{q}_1}+Q_g^1=Q_m^1;\quad \frac{\partial E_A}{\partial\ddot{q}_2}+Q_g^2=Q_m^2;\quad \frac{\partial E_A}{\partial\ddot{q}_3}+Q_g^3=Q_m^3;\quad Q_g^i={}^0J_i^T\cdot\left\{\begin{array}{c}\sum_{j=i}^{n}M_j\cdot\bar{g}^T \\ \left[\sum_{j=i}^{n}(\bar{r}_{Cj}-\bar{p}_3)\times M_j\cdot\bar{g}\right]^T\end{array}\right\}^T \quad (39)$$

In the above formulas, $\partial E_A/\partial\ddot{q}_i=Q_{i\varnothing}$ is the generalized inertia force from every joint of the 2TR robot, while Q_g^i and Q_m^i is generalized gravitational force and generalized driving force from every driving joint. The last Q_m^i is determined with the equations developed in the [4], using the values for Q_g^i and $Q_{i\varnothing}$, previously calculated, that is presented in the following:

$$\begin{pmatrix} Q_m^1 \\ Q_m^2 \\ Q_m^3 \end{pmatrix} = \begin{bmatrix} (M_1+M_2+M_3)\cdot\ddot{q}_1-M_3\cdot a_2\cdot sq_3\cdot\ddot{q}_3-M_3\cdot a_2\cdot\dot{q}_3^2\cdot cq_3 \\ (M_2+M_3)\cdot(\ddot{q}_2+g) \\ M_3\cdot a_2^2\cdot\ddot{q}_3-M_3\cdot a_2\cdot sq_3\cdot\ddot{q}_1+{}^3I_z\cdot\ddot{q}_3 \end{bmatrix} \quad (40)$$

Matrix expression (40), dignifies generalized driving forces that characterize the dynamic behavior of 2TR robot structure.

5. CONCLUSIONS

This paper has been devoted to present a few important new formulations about the kinematical control functions and dynamic equations for a robotic system. Based on the formulation within this paper, a fundamental notion in the robot dynamics was analyzed. Using matrix exponentials, the acceleration energy has been developed and applied for the case of a 2TR robot structure. The acceleration energy represents an important dynamic notion that leads to the determination of the dynamics equations typical of the mechanical robot structure taken into study. The explicit form can be applied in case of any rigid body regardless of motion performed by the kinetic link. The acceleration energy represents an important dynamic notion that can be used in order to determine the generalized inertia forces and dynamics equations for holonomic or non holonomic robotic systems, with rigid or elastic structure.

6. REFERENCES

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