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DESIGN OPTIMIZATION PROCESS WITH APPLICATION TO THE COMPOSITE STRUCTURAL ELEMENT

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Abstract: *The paper deals with a numerical approach of modelling of laminate plates and with their optimal design. It provides bases for the modelling of mechanical behaviour of laminates by reviewing general assumptions of classical laminate theory (CLT)). Elements of optimization of laminate plates are also discussed. The thicknesses of layers with the known orientation, referred as the thickness variables, will be used as design variables. The optimization problem with strength constraints will be formulated to minimize the laminate weight. Analytical and numerical approaches outlined in this paper are accompanied by computer generated example. There are depicted distributions of numerical results during the optimization process.*
Keywords: *laminate plate,sizing optimization problem, the Modified Feasible Direction method, the Sequential Linear Programming method*

1. INTRODUCTION

The rapid growth in the use of composite materials in structures has required the development of the theory of mechanics of composite laminates and the analysis and optimization of structural elements made of composite laminates. In this paper there are included an explanation of the concepts involved in the analysis and optimization of laminates, the mechanics needed to translate those concepts into a mathematical representation of the physical reality, and a explanation of the solution of the resulting boundary value problems by using Finite Element Analysis software.

2. MODELLING AND ANALYSIS OF LAMINATE PLATES, CLASSICAL LAMINATE THEORY

In the classical laminate theory the Kirchhoff hypotheses of the classical plate theory remains valid [1-3].
These assumptions imply that the transverse displacement w is independent of the thickness coordinate z, the strains γ_{xz}, γ_{yz} and ε_z are zero and the curvatures κ_i are given by

$$\boldsymbol{\kappa} = -\left(\frac{\partial\psi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial x}\right) \tag{1}$$

The equilibrium equations will be formulated for a plate element $dxdy$ and yield three force and two moment equation

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{yx}}{\partial y} = -p_1 \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = -p_2 \qquad \frac{\partial V_{xz}}{\partial x} + \frac{\partial V_{yz}}{\partial y} = -p_3$$
$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = V_{xz} \qquad \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} = V_{yz} \tag{2}$$

The transverse shear force resultants V_{xz}, V_{yz} can be eliminated and the five equations (2) reduce to three equations. The in-plane force resultants N_x, N_y and N_{xy} are uncoupled with the moment resultants M_x, M_y and M_{xy}. The three equilibrium equations are

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{yx}}{\partial y}=-p_1 \qquad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=-p_2 \qquad \frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x \partial y}+\frac{\partial^2 M_y}{\partial y^2}=-p_3 \tag{3}$$

The equations are independent of material laws and present the static equations for the undeformed plate element. In-plane reactions can be caused by coupling effects of unsymmetric laminates or sandwich plates.
Putting the constitutive equations

$$\begin{pmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{pmatrix}=\begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{pmatrix}\begin{pmatrix} \bar{\boldsymbol{\varepsilon}} \\ \boldsymbol{\kappa} \end{pmatrix} \tag{4}$$

into the equilibrium (3) and replacing using the in-plane strains $\boldsymbol{\varepsilon}$ and the curvatures $\boldsymbol{\kappa}$ by

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}=\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \end{pmatrix} \qquad \begin{pmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}=-\begin{pmatrix} \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ 2\frac{\partial^2 w}{\partial x \partial y} \end{pmatrix} \tag{5}$$

we get the differential equations for general laminate plates [2].

3. DESIGN OPTIMIZATION AND SENSITIVITY ANALYSIS

Design optimization refers to the automated redesign process that attempts to minimize or maximize objective function subject to limits or constraints on the response by using a rational mathematical approach to yield improved designs. A feasible design is a design that satisfies all of the constraints. A feasible design may not be optimal. An optimum design is defined as a point in the design space for which the objective function is minimized or maximized and the design is feasible. If relative minima exist in the design space, other optimal designs can exist.

3.1. Numerical aspects of optimization process

The basic problem is the minimization of a function subject to inequality constraints.

$$\begin{aligned} & Z = F(\boldsymbol{X}) \rightarrow \min \\ & \overline{X}_i^L \le X_i \le \overline{X}_i^U \qquad i = 1,2, ..., N_d \\ & g_j(\boldsymbol{X}) \le 0 \qquad j = 1,2, ..., N_c \end{aligned} \tag{6}$$

where $\boldsymbol{X}$ is design variable.
Linear, quadratic, cubic, or quadratic cross-terms may be selected for the polynomial approximation depending on the approximation type. They are as follows

$$F = a_0 + \sum_{i=1}^{N_d} a_i X_i + \sum_{i=1}^{N_d} b_i X_i^2 + \sum_{i=1}^{N_d-1} \sum_{j=i-1}^{N_d} c_{ij} X_i X_j + \sum_{i=1}^{N_d} d_i X_i^3 \tag{7}$$

where:
N_d is number of design variables,
X_i is ith design variable,
a_i, b_i, c_{ij}, d_i are coefficients to be determined.

Singular value decomposition (SVD) is used for regression analysis.
After the objective function and constraints are approximated and their gradients with respect to the design variables are calculated based on the approximation, we are able to solve the approximate optimization problem. One of the algorithms used in the optimization module is called the Modified Feasible Direction method (MFD). Using the Modified Feasible Direction method (MFD) [4] the solving process is iterated until convergence is achieved:

1. $q = 0$, $X^q = X^m$.
2. $q = q+1$.
3. Evaluate objective function and constraints.
4. Identify critical and potentially critical constraints $\overline{N}_c$.
5. Calculate gradient of objective function $\nabla F(X_i)$ and constraints $\nabla g_k(X_i)$, where $k = 1,2,..., \overline{N}_c$.

6. Find a usable-feasible search direction S^q.
7. Perform a one-dimensional search $X^q = X^{q-1} + \alpha S^q$.
8. Check convergence. If satisfied, make $X^{m+1} = X^q$. Otherwise, go to 2.
9. $X^{m+1} = X^q$.

Within the Kuhn-Tucker conditions the Lagrangian multiplier method was used.
By using the Lagrangian multiplier method, we define the Lagrangian function as the following

$$L = F(X_1,...,X_n) + \sum_{j=1}^{k} \lambda_j h_j(X_1,...,X_n) + \sum_{j=1}^{m} \mu_j \left[g_j(X_1,...,X_n) + s_j^2 \right] \tag{8}$$

where λ_j, j =1, ...,k and μ_j, j=1, ..., m are Lagrangian multiplicators and s_j is a slack variable which measures how far the j^{th} constraint is from being critical.
Differentiating the Lagrangian function with respect to all variables we obtain the Kuhn-Tucker conditions which are summarized as follows

$$\frac{\partial F}{\partial X_i} + \sum_{j=1}^{k} \lambda_j \frac{\partial h_j}{\partial X_i} + \sum_{j=1}^{m} \mu_j \frac{\partial g_j}{\partial X_i} = 0, \qquad i = 1, ..., n \tag{9}$$

Stationarity with respect to λ_j, j = 1, ... ,k gives following restrictions

$$h_j(X_1, ..., X_n) = 0, \qquad j = 1, ..., k. \tag{10}$$

Stationarity L with respect to s_j, gives $\mu_j s_j = 0$ and $\partial^2 L / \partial s_j^2$ for minimum of F implicates $\mu_j \geq 0$, j= 1,..., m.
Finally we get the following equations:

$$\mu_j = 0, \quad \text{if} \quad g_j(X_1, ..., X_n) < 0, \qquad j=1, ..., m$$
$$\mu_j \geq 0, \quad \text{if} \quad g_j(X_1, ..., X_n) = 0 \tag{11}$$

The physical interpretation of these conditions is that the sum of the gradient of the objective function and the scalars λ_j times the associated gradients of the active constraints must vectorally add to zero as shown in Figure 1.

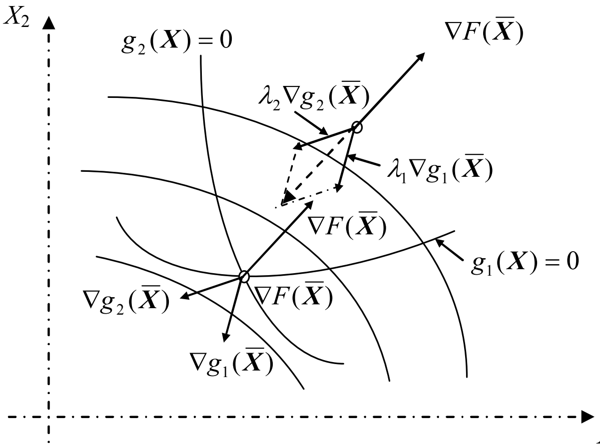

Figure 1: Kuhn-Tucker conditions at a constrained optimum

The Kuhn-Tucker conditions are also sufficient for optimality when the number of active constraints is equal to the number of design variables. Otherwise, sufficient conditions require the second derivatives of the objective function and constraints (Hessian matrix) similar to the unconstrained one. If the objective function and all of the constraints are convex, the Kuhn-Tucker conditions are also sufficient for global optimality.

4. MINIMUM WEIGHT OF A LAMINATE PLATE SUBJECT TO STRENGTH CONSTRAINT

Design of a laminate plate (Fig. 2) with orientation of angle $[0/45/\text{-}45/90]_S$ under loading N_x = 725.6 kN/m, N_y = 181.4 kN/m. Properties of the layers correspond to that of AS4/3501-6 Carbon/Epoxy material. The maximum strain failure limits for the material are $\varepsilon_1^t = \varepsilon_1^c = 0.0115$, $\varepsilon_2^t = \varepsilon_2^c = 0.00535$ and $\gamma_{12}^s = 0.02$.

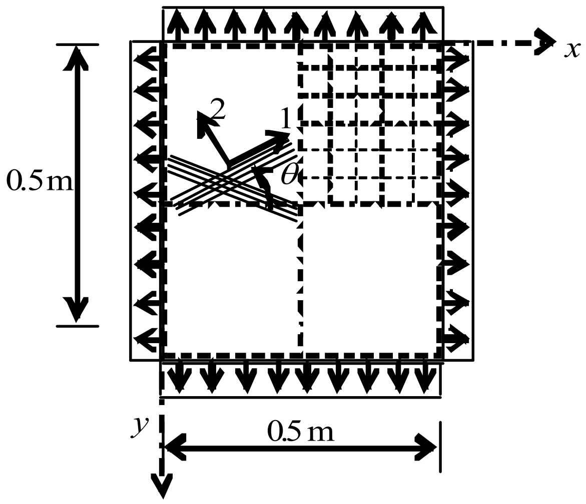

Figure 2: Problem of geometry

Table 1: Design set number: 1

Design Variable	Value
1	0.2000000E-03
2	0.2000000E-03
3	0.2000000E-03
4	0.2000000E-03

Objective	F. E. Value
Function	0.8400004E+01

Table 2: Design set number : 44

Design Variable	Value
1	0.3521971E-03
2	0.1680379E-03
3	0.1696104E-03
4	0.4895812E-04

Objective	F. E. Value
Function	0.7757434E+01

We consider the optimal design of a symmetric laminate with fixed orientation of angles. Because of the laminate symmetry, only the thicknesses $t_k, k = 1,..., I$, of one-half of the total number of layers, $I = N/2$, are used as design variables. The laminate is considered to be under the action of combined uniform in-plane stress resultants N_x and N_y.
The optimization problem is formulated in the following form

minimize $$W = \sum_{k=1}^{I} 2\rho_k t_k \quad (12)$$

subject to $$g_{kj} = \left(P_j^{(k)}\varepsilon_{1k} + Q_j^{(k)}\varepsilon_{2k} + R_j^{(k)}\gamma_{12k}\right) - 1 \le 0 \quad (13)$$

for $k = 1,\ldots,I$, $j = 1,\ldots,J$,

where ρ_k and t_k are the density and the thickness, respectively, of the k^{th} layer, $P_j^{(k)}$, $Q_j^{(k)}$, $R_j^{(k)}$, are coefficients that define the j^{th} boundary of a failure envelope for each layer in the strain space, and the $\varepsilon_{1k}, \varepsilon_{2k}, \gamma_{12k}$ are the strains in the principal material direction in the k^{th} layer. For a maximum strain criterion, which puts bounds on the values of the strains in the principal material directions, the failure envelope has four facets with P and Q defined as a inverse of the normal failure strains in the longitudinal and transverse directions to the fibers, once in tension and once compression. The coefficient R is the inverse of the shear failure strain for positive shear and for negative shear. The nonlinear programming problem is transformed to a linear by the help of sequential linear programming. The strength constraint of Eq. (13) is a nonlinear function of the thickness variables and, therefore, is linearized as

$$g_{kjL}(t_k) = g_{kj}(t_{k0}) + \sum_{i=1}^{I}(t_i - t_{0i})\left(P_j^{(k)}\frac{\partial \varepsilon_{1k}}{\partial t_i} + Q_j^{(k)}\frac{\partial \varepsilon_{2k}}{\partial t_i} + R_j^{(k)}\frac{\partial \gamma_{12k}}{\partial t_i}\right) \quad (14)$$

where $\partial\varepsilon_{1k}/\partial t_i$, $\partial\varepsilon_{2k}/\partial t_i$, $\partial\gamma_{12}/\partial t_i$ are the derivatives of the principal material direction strains in the k^{th} layer with respect to the thickness of the i^{th} layer. For a specified in-plane loading, the derivative of the laminate strains with respect to the thickness variables can be determined by differentiating in-plane part of stress-strain relation

$$\frac{\partial \boldsymbol{N}}{\partial t_i} = \frac{\partial \boldsymbol{A}}{\partial t_i}\boldsymbol{\varepsilon}^0 + \boldsymbol{A}\frac{\partial \boldsymbol{\varepsilon}^0}{\partial t_i} = 0 \tag{15}$$

The derivatives of the mid-plane strains are

$$\frac{\partial \boldsymbol{\varepsilon}^0}{\partial t_i} = -A^{-1}\overline{\boldsymbol{Q}}_{(i)}\boldsymbol{\varepsilon}^0 \tag{16}$$

where

$$\frac{\partial \boldsymbol{A}}{\partial t_i} = \overline{\boldsymbol{Q}}_{(i)} \tag{17}$$

The derivatives of the strains in the fiber and transverse to the fiber are calculated from

$$\frac{\partial \boldsymbol{\varepsilon}_{(k)}}{\partial t_i} = \boldsymbol{T}_{(k)}\frac{\partial \boldsymbol{\varepsilon}^0}{\partial t_i} \tag{18}$$

the linear approximations to the strain constraints can be constructed using Eq. (13) at any step of the sequential linearizations.

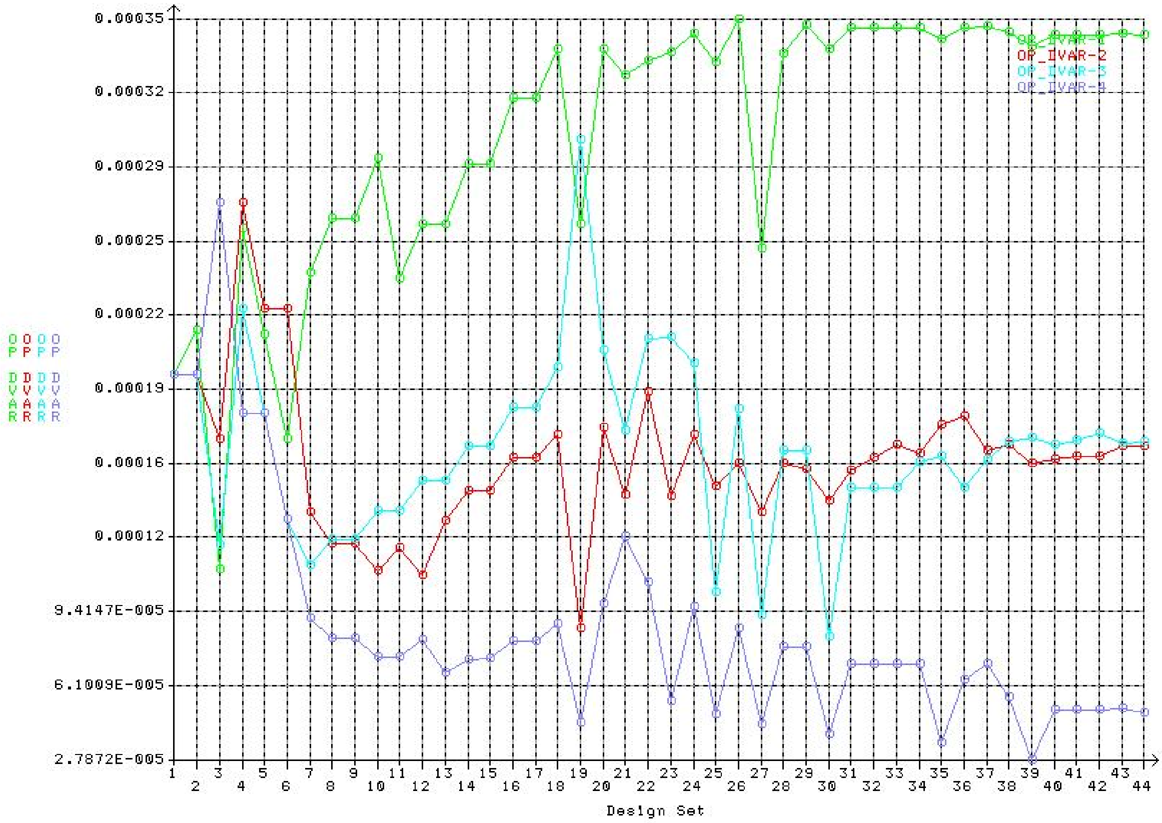

Figure 3: Design variables during the optimization process

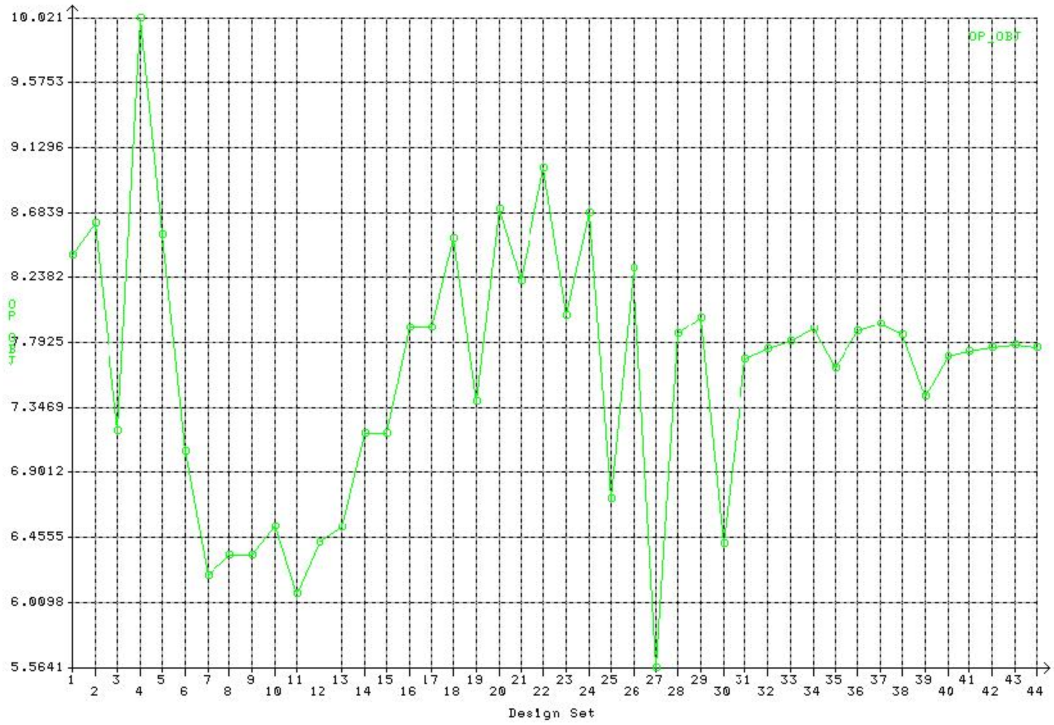

Figure 4: Objective function during the optimization process

5. CONCLUSION

The example combines describing of laminate plate modelling with optimization techniques that enable to find the best design.
For the modelling and analysis we used the classical laminate theory. We assumed the assumptions according to the Kirchhoff´s classical plate theory.
The general optimization contains [5, 6]:

1. Initial analysis with input dates.
2. Mathematical optimization problem as follows (Eqs. 12-18).
3. Linear approximation of objective function and constraints.
4. Own algorithm of MFD method with convergence criteria.
5. Convergence or termination checks of general optimization.

The maximum number of MFD iterations was 100. The general optimization process was stopped after 44 design sets (Figs. 3, 4), because the difference between the current value and the one or two previous designs was less than tolerance. Results of the optimization process following (Figs. 3,4) are listed in the Table 2.
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