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SOLUTION OF THE THERMOELASTIC EQUILIBRIUM PROBLEM FOR CYLINDRICAL TUBES WITH BIG TORSION ANGLE IN THE CASE OF COMPOSITE MATERIAL STRUCTURES

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Abstract: *In this paper, the coupled problem of the thermoelastic equilibrium for cylindrical tubes with big torsion angle for composite materials is solved. The mathematical model of isotropic nanoelastic (auxetic) cylinders with big torsion angle is also presented. Using this new mathematical model, we actually solve the thermoelasticity problem for cylindrical tubes.*
Keywords: *thermoelasticity, elastic tube, torsion angle, state of stress.*

Consider an isotropic homogeneous cylindrical nanoelastic bar for composite materials. The lateral surface is external forces free. Body forces are absent. Assume that the section is a bounded (simply or multiply connected) domain. We take the origin of coordinates at the centroid of the end section B_1 ($z=0$), Oz axes parallel to the generators and Ox, Oy axes arbitrarily directed. The point B_2 is obtained for $z=L$, where L is the length of the bar (sufficiently long). The ends are acted on by distributed forces reduced to twisting moments M of opposite sense $\overrightarrow{M}_3^0(B_1)=-\overrightarrow{M}_3^0(B_2)=M\vec{k}$. This is the classical model of the torsion [1], [3], [5], [7], [9].

In this paper, we investigate the homogenized problem of the stationary thermoelasticity of a cylindrical tube with a big torsion angle, the deformation and the state of stress being caused by the torsion moment M (which is known) and by the heating of the boundary or by the change of temperature between the boundary and the environment.

1. THE MATHEMATICAL MODEL

Here, we present A. Y. Ishlinski's mathematical model (see [4]) for the torsion of a cylindrical bar in the case of nanoelastic composite materials with a big torsion angle.

Consider now an element of length l located at distance r from the Oz-axis. Let α denote the specific torsion angle. After torsion, the generators of the cylindrical tube take the shape of a circular propeller of length

$$ds'=\sqrt{l^2+r^2\alpha^2}\,. \tag{1}$$

If the expression $\frac{r\alpha}{l}$ is sufficiently small, the specific elongation is

$$\frac{1}{l}\left(\sqrt{l^2+r^2\alpha^2}-l\right)\cong\frac{r^2\alpha^2}{2l^2}\,. \tag{2}$$

For a generator located at distance r from the cylindrical axis, we denote by χ the specific torsion coefficient:

$$\frac{r^2\alpha^2}{2l^2}=\chi r^2\text{, where }\chi=\frac{\alpha^2}{2l^2}\,. \tag{3}$$

We take the expression χr^2 as a measure of the specific elongation to Oz-axis, therefore it will represent the component ε_{zz} of the deformation tensor. In cylindrical coordinates (r,φ,z), the nonzero deformations are (see [4]):

$$\varepsilon_{zz} = \chi r^2 ,\ \varepsilon_{rr} = \frac{\partial u}{\partial r} ,\ \varepsilon_{\varphi\varphi} = \frac{u}{r} . \tag{4}$$

In (4), $u_r = u$ represents the radial displacement. Due to the condition of uniform torsion, the tangential displacement u_φ will be constant for any constant radius r and will not occur in the components of the deformation tensor.

The constitutive equations in the case of linear isotropic thermoelasticity, which generalize Cauchy and Hooke's equations, are

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu\varepsilon_{ij} - \beta T\delta_{ij} ,$$

$$\varepsilon_{ij} = \frac{1+\nu}{E} T_{ij} - \frac{\nu}{E}\Theta\delta_{ij} + \overline{\alpha} T\delta_{ij} ,$$

$$\theta - 3\overline{\alpha} = \frac{1-2\nu}{E}\Theta ,$$

$$\beta = \frac{E}{1-2\nu}\overline{\alpha} ,$$

and they are due to Duhamel and Neumann (see [2], [8]). Here, λ and μ are Lamé's coefficients, E is Young's modulus, ν is Poisson's coefficient (see [1]), T is the temperature, $\overline{\alpha}$ is the coefficient of linear dilatation, $\theta = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$, $\Theta = T_{11} + T_{22} + T_{33}$.

Due to the axial symmetry with respect to the Oz-axis and to the conditions which depends only on r, in cylindrical coordinates we have

$$\begin{aligned} T_{zz} &= \lambda(\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi}) + 2\mu\varepsilon_{zz} - \beta T(r), \\ T_{rr} &= \lambda(\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi}) + 2\mu\varepsilon_{rr} - \beta T(r), \\ T_{\varphi\varphi} &= \lambda(\varepsilon_{zz} + \varepsilon_{rr} + \varepsilon_{\varphi\varphi}) + 2\mu\varepsilon_{\varphi\varphi} - \beta T(r), \\ T_{rz} &= T_{r\varphi} = T_{z\varphi} = 0. \end{aligned} \tag{5}$$

Substituting (4) into (5) we get

$$\begin{aligned} T_{zz} &= (\lambda + 2\mu)\chi r^2 + \lambda\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) - \beta T(r), \\ T_{rr} &= (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\left(\chi r^2 + \frac{u}{r}\right) - \beta T(r), \\ T_{\varphi\varphi} &= (\lambda + 2\mu)\frac{u}{r} + \lambda\left(\chi r^2 + \frac{\partial u}{\partial r}\right) - \beta T(r). \end{aligned} \tag{6}$$

The equilibrium equation is

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\varphi\varphi}}{r} = 0 , \tag{7}$$

By substituting (6) into (7), we get the linear differential equation of the radial displacement $u_r = u(r)$:

$$u'' + \frac{1}{r}u' - \frac{1}{r^2}u = -\frac{2\nu}{1-\nu}\chi r + \frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\beta T'(r). \tag{8}$$

The torsion problems without heating for the cylinder and for the cylindrical tube, respectively, are developed in [4] and [6].

2. THE SOLUTION OF THE THERMOELASTIC EQUILIBRIUM PROBLEM

First, we solve the problem of the temperature distribution $T(r)$ for the annulus $a \le r \le b$. Due to the axial symmetry and to the boundary conditions (independent on φ), the heat equation will be

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = 0 ,\ a \le r \le b . \tag{9}$$

Theorem 1. If we associate to equation (9) the boundary conditions for $T(r)$ compatible in an exclusive manner, without heat sources, we get the situations (I-V) below, with the corresponding solutions:

(I) $\begin{cases} T(a)=0 \\ T(b)=T^* \end{cases} \rightarrow T(r)=\dfrac{T^*}{\ln\dfrac{b}{a}}\ln r-\dfrac{T^*\ln a}{\ln\dfrac{b}{a}}$,

(II) $\begin{cases} T(a)=T^* \\ T(b)=0 \end{cases} \rightarrow T(r)=\dfrac{T^*}{\ln\dfrac{a}{b}}\ln r-\dfrac{T^*\ln b}{\ln\dfrac{a}{b}}$,

(III) $\begin{cases} T(a)=0 \\ \dfrac{\partial T}{\partial r}(b)=T^* \end{cases} \rightarrow T(r)=T^* b\ln r-T^* b\ln a$,

(IV) $\begin{cases} T(a)=T^* \\ \dfrac{\partial T}{\partial r}(b)=0 \end{cases} \rightarrow T(r)=T^* a\ln r-T^* a\ln b$,

(V) $\begin{cases} T(a)=T_1 \\ T(b)=T_2 \end{cases} \rightarrow T(r)=k\ln r+k_1$,

where $k=\dfrac{T_2-T_1}{\ln(b/a)}$, $k_1=\dfrac{T_1\ln b-T_2\ln a}{\ln(b/a)}$. Moreover, considering the change of temperature between the boundary and the environment, we have the general conditions

$m_i T(r_i)+n_i\dfrac{\partial T}{\partial r}(r_i)=p_i$, $i=1,2$,

where $r_1=a$, $r_2=b$, with similar types of solutions as (I-V).

We denote by $T(r)=k\ln r+k_1$ the solution $T(r)$ corresponding to any case (I-V). Therefore, $T'(r)=\dfrac{k}{r}$ in all these cases. Consequently, equation (8) becomes

$$r^2u''+ru'-u=-\frac{2\nu}{1-\nu}\chi r^3+\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\beta kr,$$

and its general solution will be

$$u(r)=c_1 r+c_2\frac{1}{r}-\frac{\nu}{4(1-\nu)}\chi r^3+\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}\beta kr. \qquad (10)$$

Theorem 1. The solution of the thermoelastic equilibrium problem for cylindrical tubes with big torsion angle is

$$u(r)=-\frac{\nu\chi}{4(1-\nu)}\left[r^3+(1-2\nu)\left(a^2+b^2\right)r+\frac{a^2b^2}{r}\right]+\frac{(1+\nu)(1-2\nu)}{2E}\beta r\left[\frac{1-2\nu}{1-\nu}k\frac{b^2\ln b-a^2\ln a}{b^2-a^2}-k+2k_1\right]$$
$$+\frac{(1+\nu)(1-2\nu)}{2E(1-\nu)}\beta k\frac{a^2b^2(\ln b-\ln a)}{b^2-a^2}\frac{1}{r}+\frac{(1+\nu)(1-2\nu)}{2E(1-\nu)}\beta kr\ln r,$$

$$T_{rr}=\frac{E\nu\chi}{4\left(1-\nu^2\right)}\left[r^2-\left(a^2+b^2\right)+\frac{a^2b^2}{r^2}\right]+\frac{1-2\nu}{2(1-\nu)}\beta k\left[\frac{b^2\ln b-a^2\ln a}{b^2-a^2}-\frac{a^2b^2(\ln b-\ln a)}{b^2-a^2}\frac{1}{r}-\ln r\right],$$

$$T_{zz}=\frac{E\chi}{1-\nu^2}\left[r^2-\frac{a^2+b^2}{2}\nu^2\right]-\frac{1-2\nu}{1-\nu}\beta k\ln r+\frac{\nu(1-2\nu)}{1-\nu}\beta k\frac{b^2\ln b-a^2\ln a}{b^2-a^2}-\frac{\nu(1-2\nu)}{2(1-\nu)}\beta k-(1-2\nu)\beta k_1,$$

$$T_{\varphi\varphi}=\frac{E\nu\chi}{4\left(1-\nu^2\right)}\left[3r^2-\left(a^2+b^2\right)-\frac{a^2b^2}{r^2}\right]+\frac{1-2\nu}{2(1-\nu)}\beta k\left[\frac{b^2\ln b-a^2\ln a}{b^2-a^2}+\frac{a^2b^2(\ln b-\ln a)}{b^2-a^2}\frac{1}{r^2}-\ln r-1\right].$$

Proof. Replacing $T(r)=k\ln r+k_1$ and $u(r)$ from (10) into T_{rr} from (6) and using the boundary conditions $T_{rr}(a)=0$, $T_{rr}(b)=0$ (the lateral surface is stress free), we can find the constants c_1, c_2 and then the above formulas for $u(r)$, T_{rr}, T_{zz}, $T_{\varphi\varphi}$.

Remark. For a given moment M, the constant χ can be found from the following identity

$$M = \iint_D T_{\varphi\varphi} r d\sigma \, , \; D : a \le r \le b, 0 \le \varphi \le 2\pi \; .$$

We obtain

$$\chi = \frac{1-\nu^2}{E\nu} \frac{15}{b^5 - a^5 - 5a^2b^2(b-a)} \left\{ \frac{M}{2\pi} - \frac{1-2\nu}{2(1-\nu)} \beta k \left[\frac{4a^2b^2(\ln b - \ln a)}{3(a+b)} - \frac{2(b^3-a^3)}{9} \right] \right\}.$$

In particular, in the absence of the heat, the solutions for the cylindrical tube can be found by making $\beta = 0$ (see [6]):

$$u(r) = -\frac{\nu\chi}{4(1-\nu)} \left[r^3 + (1-2\nu)(a^2+b^2)r + \frac{a^2b^2}{r} \right],$$

$$T_{rr} = \frac{E\nu\chi}{4(1-\nu^2)} \left[r^2 - (a^2+b^2) + \frac{a^2b^2}{r^2} \right],$$

$$T_{zz} = \frac{E\chi}{1-\nu^2} \left[r^2 - \frac{a^2+b^2}{2} \nu^2 \right],$$

$$T_{\varphi\varphi} = \frac{E\nu\chi}{4(1-\nu^2)} \left[3r^2 - (a^2+b^2) - \frac{a^2b^2}{r^2} \right].$$

Let us notice that, due to the heat, the deformations, the displacements and the stresses are different in comparison to the case of a simple torsion (by the occurrence of β). Various studies and diagrams can also be done for stresses and deformations in the thermoelastic case.

In the case of a simple torsion, the new deformed radii of the new tube become

$$a' = a + u(a) = a\left[1 - \frac{\nu\chi}{2}(a^2+b^2) \right],$$

$$b' = b + u(b) = b\left[1 - \frac{\nu\chi}{2}(a^2+b^2) \right].$$

By fixing the distance L between the ends of the tube, the inner radius a and the outer radius b, then the thickness $d = b - a$ becomes after torsion

$$d' = b' - a' = d\left[1 - \frac{\nu\chi}{2}(a^2+b^2) \right],$$

with $d' < d$.

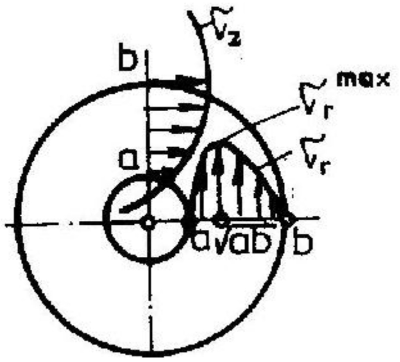

Figure 1

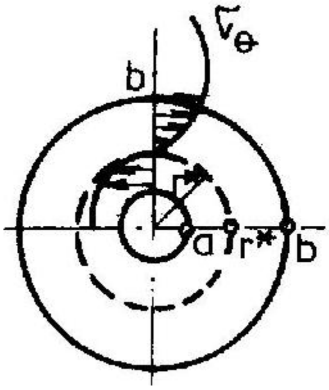

Figure 2

Diagram of the stresses: σ_r, σ_z, σ_θ

In the thermoelastic coupled case, we have

$$d' = d\left[1 - \frac{\nu\chi}{2}(a^2+b^2) + \frac{(1+\nu)(1-2\nu)}{2E} \beta \left(2k \frac{b^2 \ln b - a^2 \ln a}{b^2 - a^2} - k + 2k_1 \right) \right].$$

The variation of the thickness in the thermoelastic coupled case will depend on the terms containing β and on the inequality between T_1, T_2 (if $T(r)$ is taken from (V)).

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