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ESTIMATION OF THE VISCOELASTIC PARAMETERS OF LAMINATED COMPOSITES. PART I. ANALYTICAL CONSIDERATIONS

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Abstract : *In an effort to study the nonlinear viscoelastic behavior of neat and carbon fiber-reinforced Polyether-etherketone (PEEK) and epoxy resin, creep tests were perfomed on $[90_4]_s$ and $[\pm 45_4]_s$ laminates as well as on neat resin specimens. The laminates $[90_4]_s$ and $[\pm 45_4]_s$ are used to study the time dependent transverse and shear properties, respectively. Series of 10-hour isothermal tensile creep tests were conducted on each laminate at four temperatures (up to 140°C for the epoxy system and up to 120°C for the PEEK system) and different stress levels. An analytical procedure based on the Mori-Tanaka's mean field theory but extended into viscoelastic domain is used to account for the time dependent overall response of the composite body. Directed toward this, Schapery's nonlinear constitutive equation is input into the analysis which makes the proposed method capable of accounting for time dependent behavior of the polymeric matrix. The nonlinearity factors in the Schapery's formulation axe found based on the creep tests performed at different stress levels and test temperatures. The stress and temperature dependence of the nonlinearity factors was evaluated.using a numerical procedure based on least squares techniques. The results for the pure resin and 90 degree specimens show good agreement between experiment and the predicted data. For the 45 degree specimen, the above correlation is less impressive, nevertheless the general trend in the curves of the predicted data closely match those generated by experiment. The accuracy of the results will be more improved if the gradual change of the fiber orientation specially at high values of temperature and stress is accounted for in the analysis.*

1. INTRODUCTION

The effective viscoelastic behavior of a two phase composite body is dependent upon the elastic/viscoelastic properties of the constituent materials. A number of approaches are presented in the literature for the prediction of the bounds on elastic/viscoelastic parameters of multi (two) phase composites [1-6]. Most of the works which have been done to estimate the bounds on the effective elastic/viscoelastic property of fiber reinforced composites, assume both of the phases to possess isotropic material behavior. Little work has been presented in the literature for those cases where for instance the reinforcing phase has anisotropic or transversely isotropic properties. An example of this could be a graphite/epoxy composite where the matrix material is considered to be isotropic but the fibers show anisotropic behavior.

The problem considered for the computation of the elastic/viscoelastic moduli is that of parallel fibers which are long enough so that end effects can be neglected. The material may be represented by a cylindrical specimen whose cross section is very large in comparison to fiber cross section. The longitudinal axes of the specimen coincides with the fiber direction and since the end effects are neglected, the fibers may be assumed to run continuously through the entire length of the specimen.

It is also assumed that the specimen is statistically both homogeneous and transversely isotropic. The problem to be considered is that of predicting the bounds on elastic/- viscoelastic properties of such a specimen in terms of its geometry and the elastic/viscoelastic moduli of its constituents.

2. THEORETICAL BACKGROUND

In the following, an attempt will be made to estimate the characteristic parameters of a composite material using the Mori-Tanaka theory [7]. The method utilizes a cylindrical specimen containing elliptic cylindrical fibers in the matrix material. The fibers are considered to be transversely isotropic and elastic while the matrix is isotropic with viscoelastic properties. Two cases may be considered: one in which the fibers are monotonically aligned and uniformly dispersed which results in an orthotropic composite specimen while in the second case the fibers are randomly-oriented elliptic cylinders. For the latter case, the resulting composite is transversely isotropic. The shape of the elliptic fibers is characterized by the ratio α = *t/w* as shown in Figure 1. For a composite containing circular fibers, as in the current study, the cross sectional aspect ratio α is equal to 1.

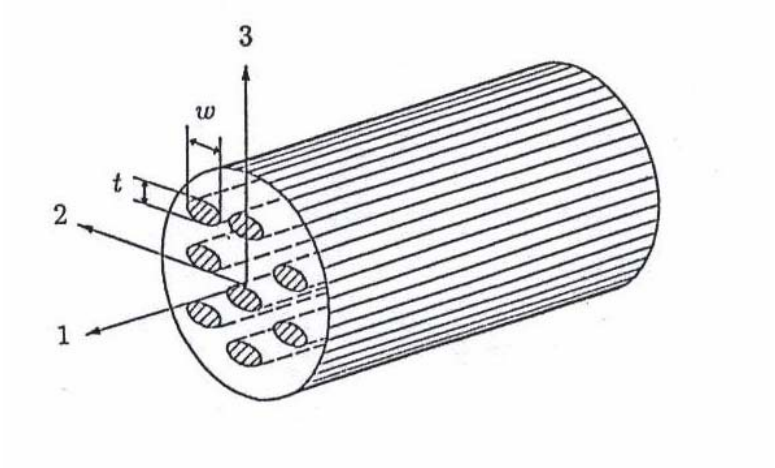

Figure 1: . Schematic representation of a composite with monotonically aligned elliptic cylindrical fibers

The solution of the problem is based on Eshelby's [8] approach for an ellipsoidal inclusion in conjunction with the Mori-Tanaka's [7] mean-field theory for which the results are obtained by Zhao and Weng [9].
It should be pointed out that in [9], the two constituents are considered to have isotropic properties. In the present investigation however, the fibers possess anisotropic material behavior. This is the author's contribution to the development of the theory to evaluate the effective moduli of a fiber reinforced composite. It is interesting to note that through this theory it is possible to obtain all the effective moduli which are required to characterize an orthotropic composite (in particular a transversely isotropic composite). In summery, using the foregoing approach, the compliance matrix of a viscoelastic body can be determined which allows one to investigate the time dependent response - here in form of creep curves.

For the understanding of the subsequent analysis, the Mori-Tanaka's theory [7] will be briefly reviewed. Let us first consider a fiber-reinforced composite in which transversely isotropic fibers are uniformly distributed in the matrix material. Here a representative volume element (RVE) of the composite and one of a comparison material (CM) made only of the matrix material are introduced. Let both of the above RVE's be subjected to the same boundary traction $\overline{\sigma}$. Let us further denote the elastic coefficients matrix of CM by C_m.
For the real composite however, under the same $\overline{\sigma}$, the mean strain in the matrix differs from that in the CM. Let $\tilde{\varepsilon}$ represent the difference of the two mean values of strain. The mean value of the stress in the CM is $\overline{\sigma}$. On the other hand, in the matrix of the composite, there exists a different mean stress $\tilde{\sigma}$. As a result, the following observations can be made:
In CM, due to the mean strain field ε° and the mean stress field $\overline{\sigma}$, the stress-strain relation becomes:

$$\overline{\sigma} = C_m \varepsilon^0 \qquad \textbf{(1)}$$

In the matrix constituent of RVE of the composite, the mean strain and stress fields are $\varepsilon^m = \varepsilon^0 + \bar{\varepsilon}$ and $\sigma^m = \bar{\sigma} + \tilde{\sigma}$ respectively. The stress-strain relation is therefore written in the following form:

$$\sigma^m = \bar{\sigma} + \tilde{\sigma} = C_m(\varepsilon^0 + \bar{\varepsilon}) \quad \textbf{(2)}$$

In the fiber of RVE of the composite, the mean strain field differs from that in matrix through an additional term ε^{pt} and hence $\varepsilon^f = \varepsilon^m + \varepsilon^{pt} = \varepsilon^0 + \tilde{\varepsilon} + \varepsilon^{pt}$. In the same manner, the mean stress field is different from that in matrix by an additional term σ^{pt} and therefore $\sigma^f = \bar{\sigma} + \tilde{\sigma} + \sigma^{pt}$. The stress-strain relation becomes:

$$\sigma^f = \bar{\sigma} + \tilde{\sigma} + \sigma^{pt} = C_f(\varepsilon^0 + \tilde{\varepsilon} + \varepsilon^{pt}) \quad \textbf{(3)}$$

It should be pointed out here that C_f is to represent the matrix of the elastic coefficients of the fiber. Using Eshelby's equivalence principle [8], one may write the average stress in fiber in terms of the elastic coefficients of the matrix C_m by introducing the term ε^* in the average strain field, i,e.

$$\sigma^f = \bar{\sigma} + \tilde{\sigma} + \sigma^{pt} = C_f(\varepsilon^0 + \tilde{\varepsilon} + \varepsilon^{pt}) = C_m(\varepsilon^0 + \tilde{\varepsilon} + \varepsilon^{pt} - \varepsilon^*) \quad \textbf{(4)}$$

where the following relation holds:

$$\varepsilon^{pt} = P\varepsilon^* \quad \textbf{(5)}$$

The four-rank tensor *P is* Eshelby's transformation tensor and has the symmetry property *Pikjl = Pjikl = Pijtk*. The components of the above tensor for the cylindrical fiber with an elliptical cross section are given in the Appendix. The average stress for the whole composite can be written as

$$\begin{aligned}\bar{\sigma} &= v_f\sigma_f + v_m\sigma_m = v_f(\bar{\sigma} + \tilde{\sigma} + \sigma^{pt}) + v_m(\bar{\sigma} + \tilde{\sigma}) \\ &= (v_f + v_m)\bar{\sigma} + (v_f + v_m)\tilde{\sigma} + v_f\sigma^{pt} = \bar{\sigma} + \tilde{\sigma} + v_f\sigma^{pt}\end{aligned} \quad \textbf{(6a)}$$

which reduces to:

$$\tilde{\sigma} = -v_f\sigma^{pt} \quad \textbf{(6b)}$$

Following a similar procedure one can obtain

$$\bar{\varepsilon} = -v_f(\varepsilon^{pt} - \varepsilon^*) = -v_f(P\varepsilon^* - \varepsilon^*) = -v_f(P - I)\varepsilon^* \quad \textbf{(7)}$$

where *I* is the unit tensor.

Substitution of this last relation into (4) yields:

$$C_f\left[\varepsilon^0 - v_f(P - I)\varepsilon^* + P\varepsilon^*\right] = C_m\left[\varepsilon^0 - v_f(P - I)\varepsilon^* + P\varepsilon^* - \varepsilon^*\right] \quad \textbf{(8a)}$$

which can be simplified to:

$$\left[C_f\left(-v_f(P - I) + P\right) + C_m\left(v_f(P - I) - P + I\right)\right]\varepsilon^* + \left(C_f - C_m\right)\varepsilon^0 = 0 \quad \textbf{(8b)}$$

This last relation can be written in an alternative form as

$$\left[C_f\left(v_m P + v_f I\right) - C_m v_m (P - I)\right]\varepsilon^* + \left(C_f - C_m\right)\varepsilon^0 = 0 \quad \textbf{(8c)}$$

or

$$\left[v_m\left(C_f - C_m\right)P + v_f\left(C_f - C_m\right) + C_m\right]\varepsilon^* + \left(C_f - C_m\right)\varepsilon^0 = 0 \quad \textbf{(8d)}$$

and finally as:

$$\left[\left(C_f - C_m\right)\left(v_m P + v_f I\right) + C_m\right]\varepsilon^* + \left(C_f - C_m\right)\varepsilon^0 = 0 \quad \textbf{(8e)}$$

From the above equation, one may write the relation between ε^* and ε^0 in the following form: (see Appendix)

$$\begin{Bmatrix} \varepsilon_{11}^* \\ \varepsilon_{22}^* \\ \varepsilon_{33}^* \end{Bmatrix} = \frac{1}{A}\begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix}\begin{Bmatrix} \varepsilon_{11}^0 \\ \varepsilon_{22}^0 \\ \varepsilon_{33}^0 \end{Bmatrix} \quad \textbf{(9)}$$

For the shear components of the strain, following relations can be written [9]:

$$\varepsilon_{12}^* = \frac{\left(G_{12,f} - G_m\right)}{\left(G_{12,f} - G_m\right)\left(2v_m P_{1212} + v_f\right) + G_m}\varepsilon_{12}^0 \quad \textbf{(10)}$$

$$\varepsilon_{23}^* = \frac{\left(G_{23,f} - G_m\right)}{\left(G_{23,f} - G_m\right)\left(2v_m P_{2323} + v_f\right) + G_m}\varepsilon_{23}^0 \quad \textbf{(11)}$$

$$\varepsilon_{31}^* = \frac{\left(G_{31,f} - G_m\right)}{\left(G_{31,f} - G_m\right)\left(2v_m P_{3131} + v_f\right) + G_m}\varepsilon_{31}^0 \quad \textbf{(12)}$$

Let us now apply the foregoing method to evaluate the elastic/viscoelastic parameters of the entire composite when it is considered as an orthotropic body. As a particular case of the above approach, the transversely isotropic composite investigated in the present study will be considered.

For the derivation of the longitudinal Young's modulus E_{11} of an orthotropic body, the composite and the comparison material are subjected to a pure tension $\bar{\sigma}_{11}$. Then, it follows that

$\bar{\sigma}_{11} = E_{11}\bar{\varepsilon}_{11}$ for the composite, and

$\bar{\sigma}_{11} = Em\bar{\varepsilon}^0_{11}$ and $\bar{\varepsilon}^0_{22} = \bar{\varepsilon}^0_{33} = -\nu_m\bar{\varepsilon}^0_{22}$ for the comparison material

Making use of the relations presented earlier, the above relations can be written in the following form:

$$\begin{aligned}\bar{\varepsilon}_{11} &= \bar{\varepsilon}^0_{11} + \nu_f\bar{\varepsilon}^*_{11} = \bar{\varepsilon}^0_{11} + \nu_f\left(\frac{A_1}{A}\bar{\varepsilon}^0_{11} + \frac{A_2}{A}\bar{\varepsilon}^0_{22} + \frac{A_3}{A}\bar{\varepsilon}^0_{33}\right)\\ &= \bar{\varepsilon}^0_{11}\left(1+\nu_f a_1\right) - \nu_f a_2\nu_m\bar{\varepsilon}^0_{11} - \nu_f a_3\nu_m\bar{\varepsilon}^0_{11}\\ &= \bar{\varepsilon}^0_{11}\left[1+\nu_f\left[a_1 - \nu_m\left(a_2+a_3\right)\right]\right]\end{aligned} \quad \textbf{(13)}$$

where the notation: $a_i = Ai/A$ has been used.

It follows that:

$$\frac{E_{11}}{E_m} = \frac{\bar{\varepsilon}^0_{11}}{\bar{\varepsilon}_{11}} = \frac{1}{1+\nu_f\left[a_1 - \nu_m\left(a_2+a_3\right)\right]} \quad \textbf{(14)}$$

Similar expressions can be obtained for the elastic moduli in the other directions

$$\frac{E_{22}}{E_m} = \frac{\bar{\varepsilon}^0_{22}}{\bar{\varepsilon}_{22}} = \frac{1}{1+\nu_f\left[a_5 - \nu_m\left(a_4+a_6\right)\right]} \quad \textbf{(15)}$$

and

$$\frac{E_{33}}{E_m} = \frac{\bar{\varepsilon}^0_{33}}{\bar{\varepsilon}_{33}} = \frac{1}{1+\nu_f\left[a_9 - \nu_m\left(a_7+a_8\right)\right]} \quad \textbf{(16)}$$

For the computation of the shear moduli, one may use the relations:

$$\bar{\sigma}_{12} = 2G_{12}\bar{\varepsilon}_{12} \qquad ; \qquad \bar{\sigma}_{12} = 2G_m\bar{\varepsilon}^0_{12} \quad \textbf{(17)}$$

Recall that:

$$\bar{\varepsilon}_{12} = \bar{\varepsilon}^0_{12} + \nu_f\bar{\varepsilon}^*_{12} \quad \textbf{(18)}$$

$$= \varepsilon^*_{12} - \nu_f\frac{G_{12,f} - G_m}{\left(G_{12,f} - G_m\right)\left(2\nu_m P_{1212} + \nu_f\right) + G_m}\varepsilon^0_{12} \quad \textbf{(19)}$$

which results in the following relation for the shear modulus G_{12}

$$\frac{G_{12}}{G_m} = 1 + \frac{v_f}{\dfrac{G_m}{G_{12,f} - G_m} + 2v_m P_{1212}} \tag{20}$$

Following a procedure similar to that just presented, the expressions for the remaining shear moduli are obtained. These are

$$\frac{G_{23}}{G_m} = 1 + \frac{v_f}{\dfrac{G_m}{G_{23,f} - G_m} + 2v_m P_{2323}} \tag{21}$$

and

$$\frac{G_{31}}{G_m} = 1 + \frac{v_f}{\dfrac{G_m}{G_{31,f} - G_m} + 2v_m P_{3131}} \tag{22}$$

Finally, in order to determine the expressions for Poisson's ratio, one can use the relation:

$$\bar{\varepsilon}_{22} = -v_m \bar{\varepsilon}_{11} \qquad ; \qquad \bar{\varepsilon}^0_{22} = \bar{\varepsilon}^0_{33} = -v_m \bar{\varepsilon}^0_{11} \tag{23}$$

Note that:

$$\begin{aligned} \bar{\varepsilon}_{11} = \bar{\varepsilon}^0_{11} + v_f \bar{\varepsilon}^*_{11} &= \bar{\varepsilon}^0_{11} + v_f a_1 \bar{\varepsilon}^0_{11} + v_f a_2 \bar{\varepsilon}^0_{22} + v_f a_3 \bar{\varepsilon}^0_{33} \\ &= \bar{\varepsilon}^0_{11}\left(1 + v_f a_1\right) + v_f a_2 \bar{\varepsilon}^0_{22} + v_f a_3 \bar{\varepsilon}^0_{33} \end{aligned} \tag{24}$$

Also

$$\bar{\varepsilon}_{22} = \bar{\varepsilon}^0_{22} + v_f \bar{\varepsilon}^*_{22} = v_f a_4 \bar{\varepsilon}^0_{11} + \bar{\varepsilon}^0_{22}\left(1 + v_f a_5\right) + v_f a_6 \bar{\varepsilon}^0_{33} \tag{25}$$

which can be written in a simplified form as:

$$\begin{aligned} \bar{\varepsilon}_{11} &= \left[\left(1 + v_f a_1\right) - v_f a_2 v_m - v_f a_3 v_m\right] \bar{\varepsilon}^0_{11} \\ &= \left[v_f a_4 - v_m\left(1 + v_f a_5\right) - v_m v_f a_6\right] \bar{\varepsilon}^0_{11} \end{aligned} \tag{26}$$

Now, substitution of the above relation in that of the *v\i* yields

$$v_{12} = -\frac{\bar{\varepsilon}_{22}}{\bar{\varepsilon}_{11}} = -\frac{v_f a_4 - v_m\left(1 + v_f a_5\right) - v_m v_f a_6}{1 + v_f a_1 - v_f a_2 v_m - v_f a_3 v_m} \quad \textbf{(27)}$$

which can be rearranged to:

$$v_{12} = \frac{v_m - v_f\left[a_4 - v_m\left(a_5 + a_6\right)\right]}{1 + v_f\left[a_1 - v_m\left(a_2 + a_3\right)\right]} \quad \textbf{(28)}$$

Similarly, it can be shown that

$$v_{23} = \frac{v_m - v_f\left[a_6 - v_m\left(a_4 + a_5\right)\right]}{1 + v_f\left[a_9 - v_m\left(a_7 + a_8\right)\right]} \quad \textbf{(29)}$$

and

$$v_{31} = \frac{v_m - v_f\left[a_7 - v_m\left(a_8 + a_9\right)\right]}{1 + v_f\left[a_1 - v_m\left(a_2 + a_3\right)\right]} \quad \textbf{(30)}$$

Note that the proposed method is in contrast to those by which only the bounds on characteristic parameters of the material are obtained. Here, the engineering constants are uniquely evaluated as a result of which, an accurate prediction of the overall material response can be made. The procedure is original in that, the fibers are not isotropic but instead possess anisotropic or transversely isotropic behavior. The components of the compliance matrix are evaluated based on the computed values of the engineering constants while time dependence is incorporated into the analysis by using Schapery's constitutive equation for nonlinear viscoelastic materials [10]. In this way, the viscoelastic response of the composite - here in form of creep curves - can be studied.

3. RESULTS AND DISCUSSION

In Figures 2 through 11, predictions made for the creep response of PEEK and epoxy resin together with those of the $[90]_{4s}$ and [±45]2s laminates are plotted. The response using the proposed theoretical technique agrees very well with the experimental data for the neat PEEK and epoxy specimens under the loading conditions indicated in Figures 2 and 3.

Examination of the plots for the transverse strain ε_{22} of the two composites (Figures 4 through 7) indicates that a very accurate prediction of the experimental results is possible by utilizing the proposed method even at relatively high temperature of 120°C. It can be seen from Figure 7 that the maximum deviation of the theoretical results from the experimental data is less than 8%.

Attention should be drawn to the fact that the [±45]2s laminate configuration can be expected to operate at a higher creep rate than a $[90]_{4s}$ laminate. The agreement between experimental and predicted data for the former laminate is less impressive in terms of accuracy.

Nevertheless the general trend in the curves of the predicted data for the stress and temperature combinations presented closely match those generated by experiment (Figures 8 through 11). The "scissoring action" which occurs in the [±45]2s laminate can lead to intolerable large creep distortion values. For this reason, it is evident that some discrepancies may occur in the prediction of the results for the [±45]2s laminates investigated in the current study. By examining the proposed method, the reason for this discrepancy becomes evident. The value for the shear modulus G12 obtained through the proposed method is based on a fixed fiber orientation of 45° in the laminate. This is however not the case in the creep test of 45° specimens especially at high values of stresses and/or test temperatures where the fiber orientation of 45°changes with time thus causing error in the prediction of the results. This can be avoided if the gradual change of the fiber orientation is accounted for in the analysis.

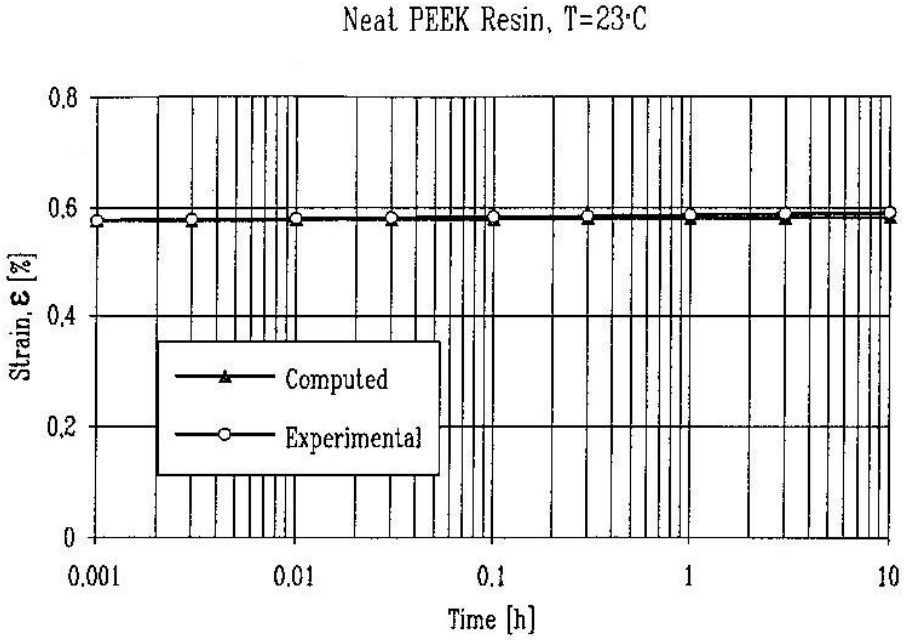

Figure2. Comparison between experimental and theoretical prediction of creep strain s for neat PEEK resin subjected to $\sigma =$ 26 MPa at 23° C.

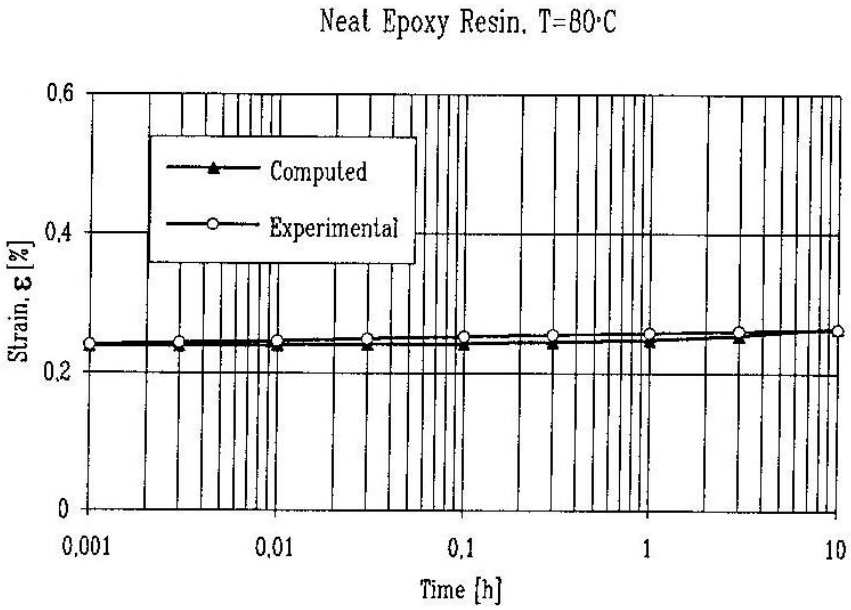

Figure3. Comparison between experimental and theoretical prediction of creep strain s for neat epoxy resin subjected to a — 8.6 MPa at 80° C.

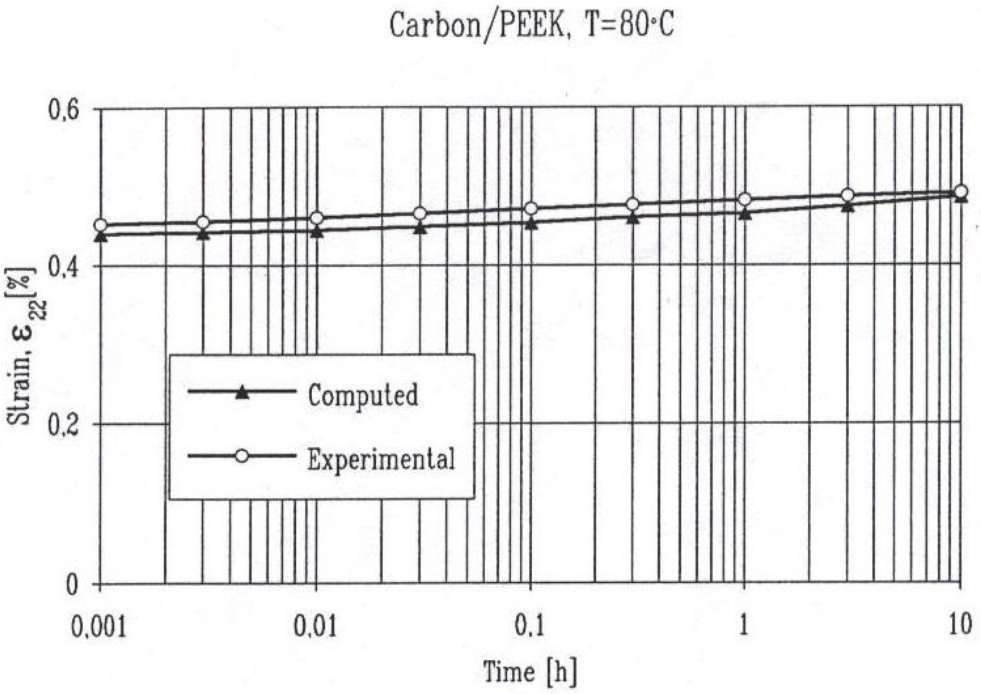

Figure 4. Comparison between experimental and theoretical prediction of creep strain ε_{22} carbon/PEEK subjected to $\sigma_{22} = 36$ MPa at 80° C.

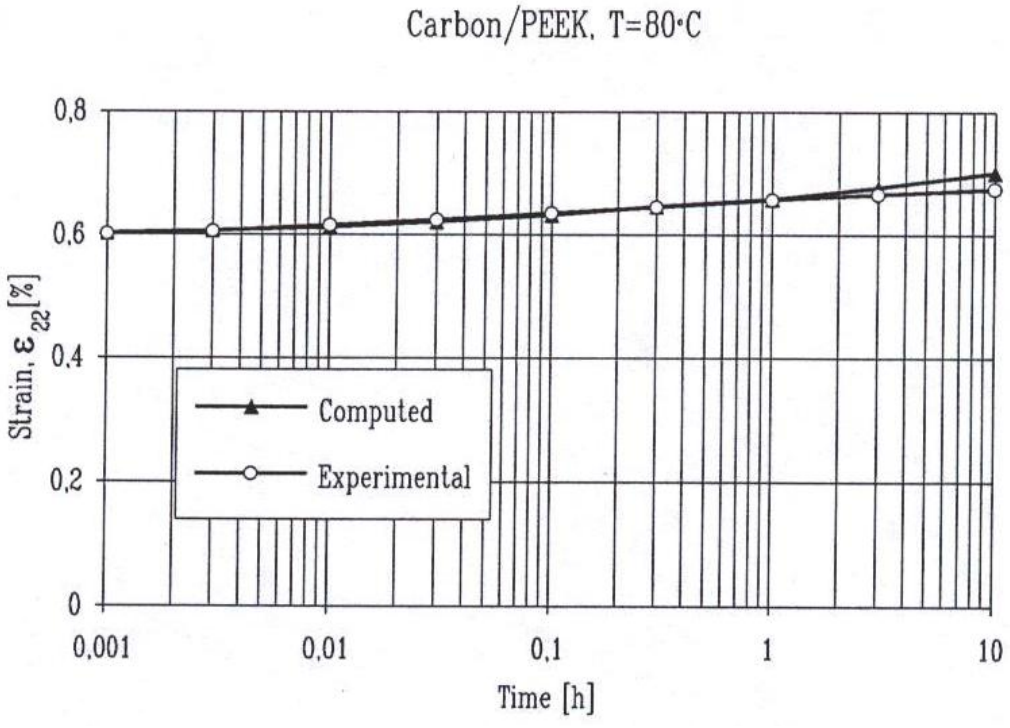

Figure 5. Comparison between experimental and theoretical prediction of creep strain ε_{22} for carbon/PEEK subjected to $\sigma_{22} = 47$ MPa at 80° C.

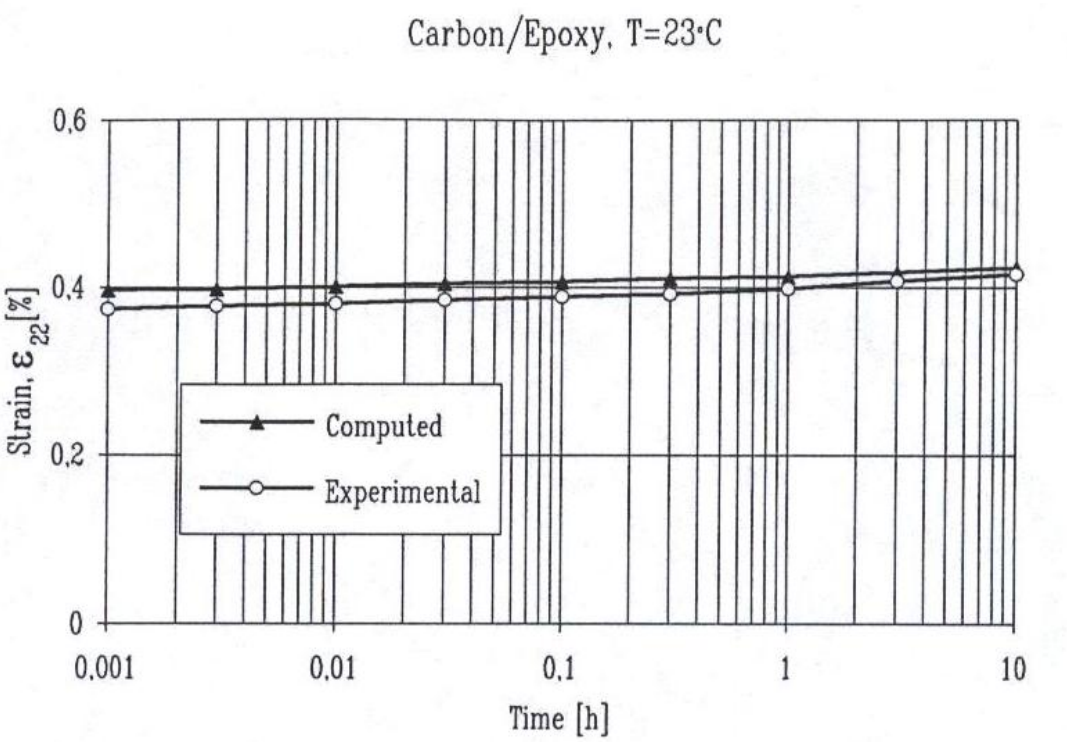

Figure 6. Comparison between experimental and theoretical prediction of creep strain ε_{22} for carbon/epoxy subjected to $\sigma_{22} = 30$ MPa at 23° C.

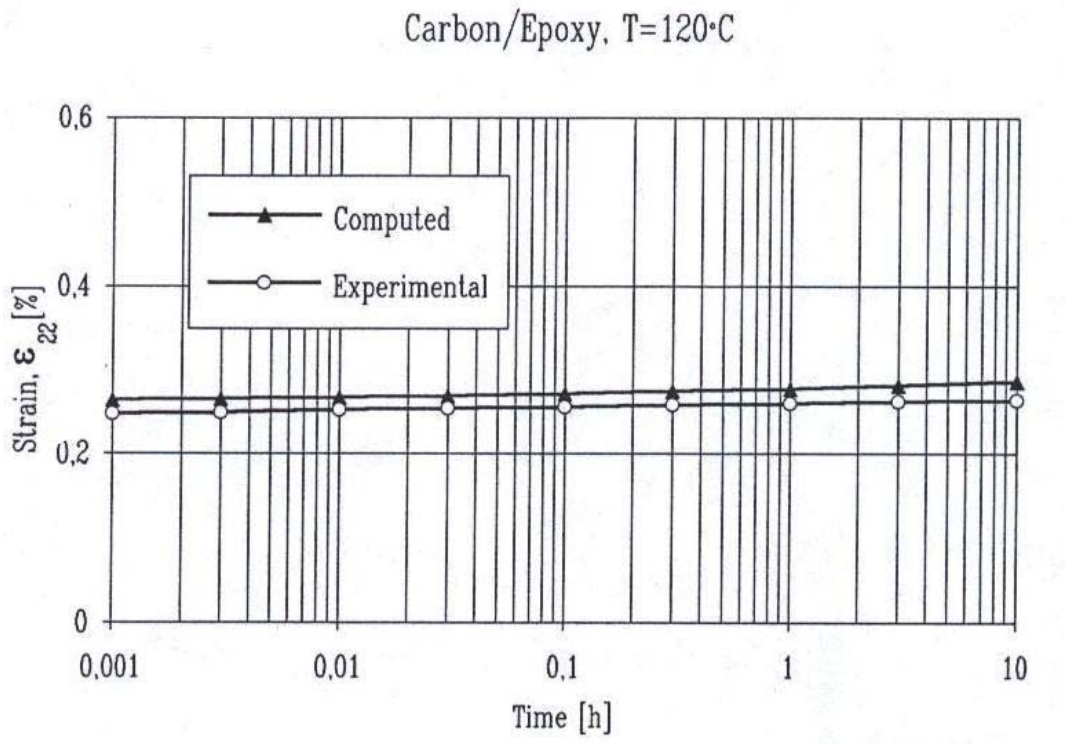

Figure 7. Comparison between experimental and theoretical prediction of creep strain ε_{22} for carbon/epoxy subjected to $\sigma_{22} = 19$ MPa at 120° C.

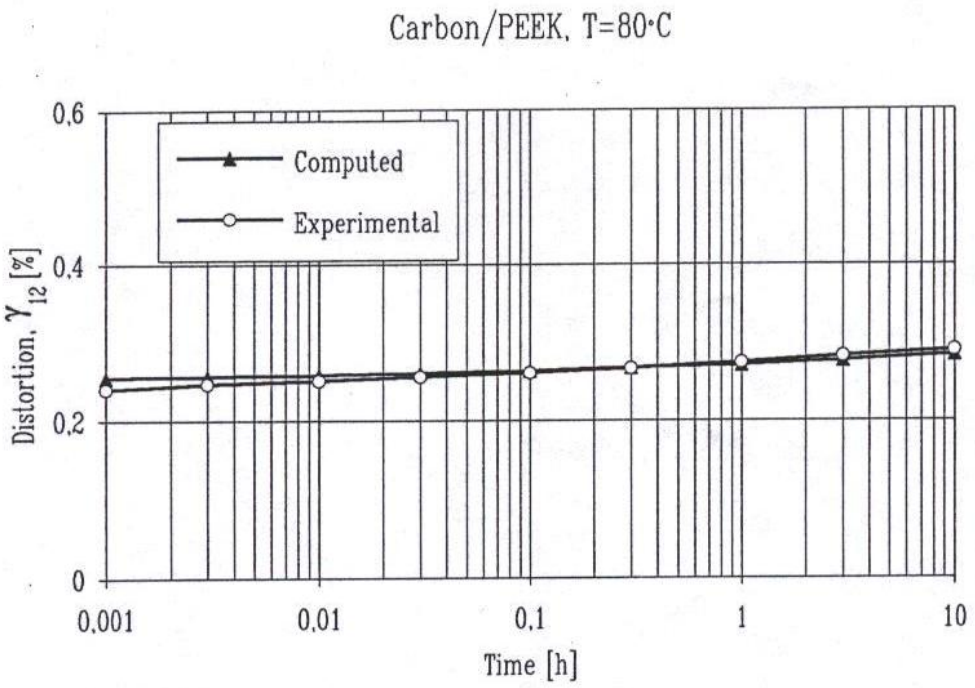

Figure 8. Comparison between experimental and theoretical prediction of creep distortion γ_{12} for carbon/PEEK subjected to $\tau_{12} = 13$ MPa at 80° C.

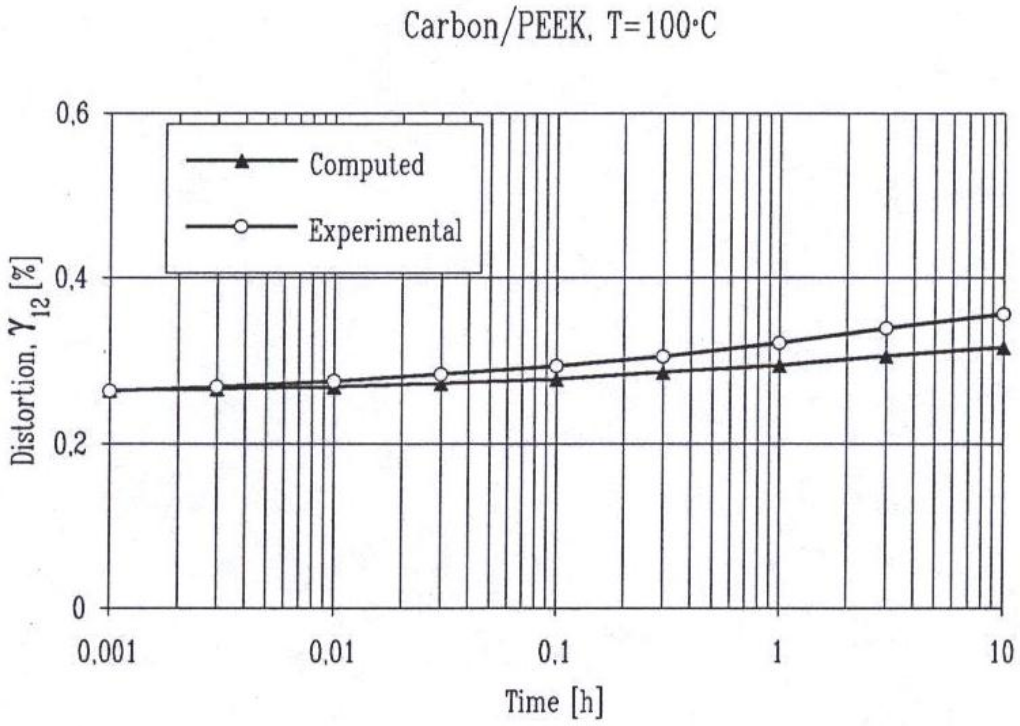

Figure 9. Comparison between experimental and theoretical prediction of creep distortion γ_{12} for carbon/PEEK subjected to $\tau_{12} = 6$ MPa at 100° C.

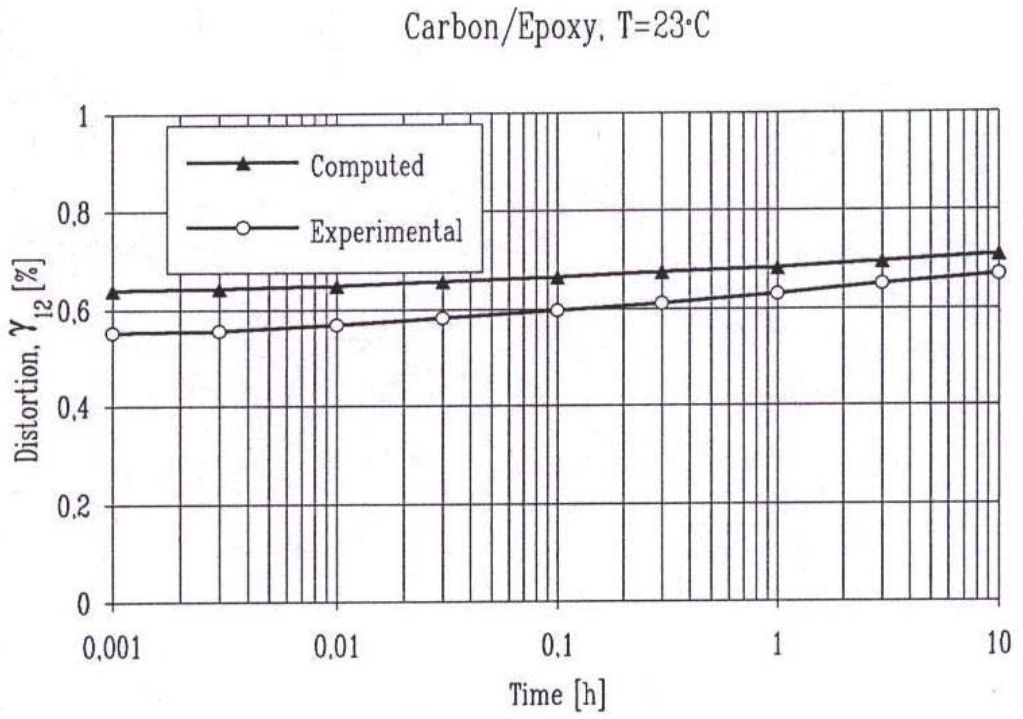

Figure 10. Comparison between experimental and theoretical prediction of creep distortion γ_{12} for carbon/epoxy subjected to $\tau_{12} = 29$ MPa at 23° C.

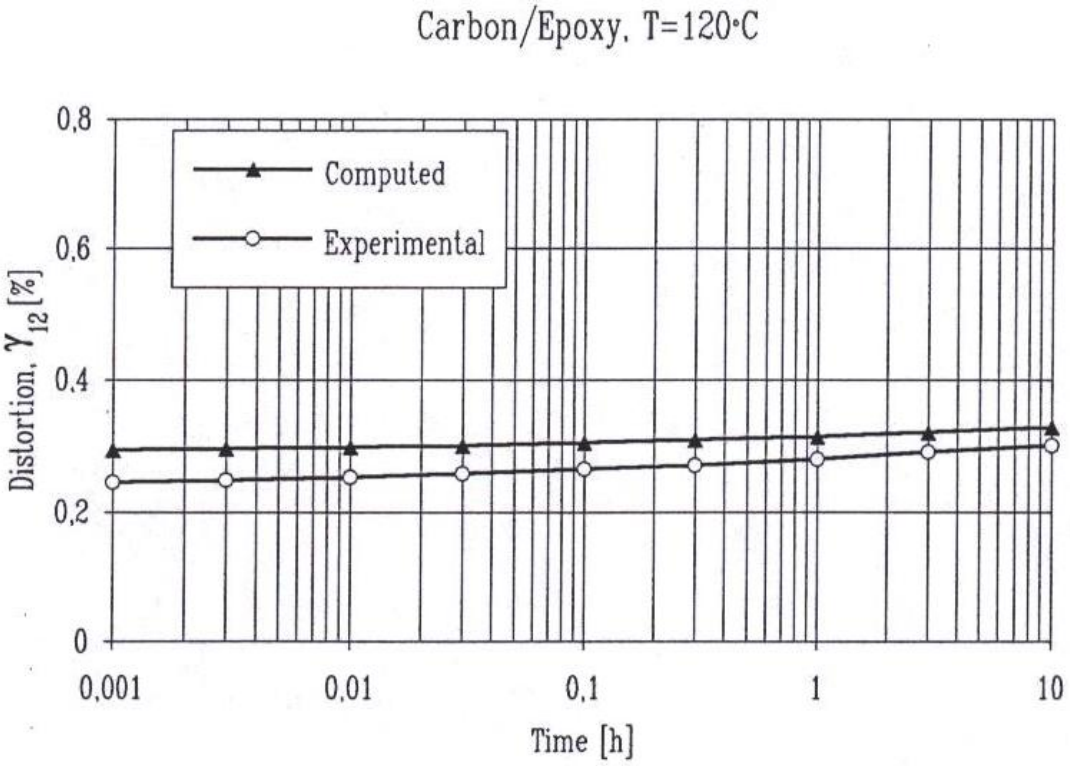

Figure 11. Comparison between experimental and theoretical prediction of creep distortion γ_{12} for carbon/epoxy subjected to $\tau_{12} = 12$ MPa at 120° C.

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APPENDIX

Eshelby's Transformation Tensor for an Elliptic Cylinder

The components of the Estielby's P_{ijkl} for the cylindrical fiber with an elliptical cross section are [11]

$$P_{2222} = \frac{1}{2(1-\nu_m)}\left[\frac{1+2\alpha}{(1+\alpha)^2} + \frac{1-2\nu_m}{1+\alpha}\right]$$

$$P_{3333} = \frac{\alpha}{2(1-\nu_m)}\left[\frac{\alpha+2}{(1+\alpha)^2} + \frac{1-2\nu_m}{1+\alpha}\right]$$

$$P_{2211} = \frac{\nu_m}{1-\nu_m}\frac{1}{1+\alpha}$$

$$P_{2233} = \frac{1}{2(1-\nu_m)}\left[\frac{1}{(1+\alpha)^2} + \frac{1-2\nu_m}{1+\alpha}\right]$$

$$P_{3311} = \frac{\nu_m}{1-\nu_m}\frac{\alpha}{1+\alpha}$$

$$P_{3322} = \frac{\alpha}{2(1-\nu_m)}\left[\frac{\alpha}{(1+\alpha)^2} + \frac{1-2\nu_m}{1+\alpha}\right]$$

$$P_{1212} = \frac{1}{2(1+\alpha)}$$

$$P_{1313} = \frac{\alpha}{2(1+\alpha)}$$

$$P_{2323} = \frac{1}{4(1-\nu_m)}\left[\frac{1+\alpha^2}{(1+\alpha)^2} + (1-2\nu_m)\right]$$

and

$$P_{1111} = 0\ ;\ P_{1122} = 0\ ;\ \ P_{1133} = 0 \qquad \textbf{(A-2)}$$

where all other $P_{ijkl} = 0$ .In all the above equations, α is the aspect ratio introduced earlier and ν_m is the Poisson's ratio of the isotropic matrix. Note also that the axis 1 is taken to be infinitely extended, axes 2 and 3 are along the thickness and along the width of the elliptic cylinder, respectively. When the fibers have circular cross section as in the present investigation, so that $\alpha = t/w \to 1$, the components of the *Pijki* are simplified accordingly.

As described earlier, substitution of Eq. (7) in Eq. (4) yields the relation between the normal components of ε_{ij}^{0} and ε_{ij}^{*} .These relations are: (see Ref. [9])

$$\begin{bmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{*} \\ \varepsilon_{22}^{*} \\ \varepsilon_{33}^{*} \end{Bmatrix} + \begin{bmatrix} N_1 & 1 & 1 \\ 1 & N_1 & 1 \\ 1 & 1 & N_1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{0} \\ \varepsilon_{22}^{0} \\ \varepsilon_{33}^{0} \end{Bmatrix} = 0 \qquad \textbf{(A-3)}$$

where

$$\begin{aligned}
M_1 &= v_f N_1 + N_2 + v_m \left(P_{2211} + P_{3311}\right) \\
M_2 &= v_f + N_3 + v_m \left(P_{2222} + P_{3322}\right) \\
M_3 &= v_f + N_3 + v_m \left(P_{2233} + P_{3333}\right) \\
M_4 &= v_f + N_3 + v_m \left(P_{2211} + P_{3311}\right) \\
M_5 &= v_f N_1 + N_2 + v_m \left(N_1 P_{2222} + P_{3322}\right) \\
M_6 &= v_f + N_3 + v_m \left(P_{2233} + P_{3333}\right) \\
M_7 &= v_f + N_3 + v_m \left(N_1 P_{3311} + P_{2211}\right) \\
M_8 &= v_f + N_3 + v_m \left(N_1 P_{3322} + P_{2222}\right) \\
M_9 &= v_f N_1 + N_2 + v_m \left(P_{3333} + P_{2233}\right)
\end{aligned} \qquad \textbf{(A-4)}$$

with the following expressions for N_i's in terms of the Lame's constants of the constituting phases

$$\begin{aligned}
N_1 &= 1 + 2\left(G_f - G_m\right)/\left(\lambda_f - \lambda_m\right) \\
N_2 &= \left(\lambda_m + 2G_m\right)/\left(\lambda_f - \lambda_m\right) \\
N_3 &= \lambda_m /\left(\lambda_f - \lambda_m\right)
\end{aligned} \qquad \textbf{(A-5)}$$

Now, in order to arrive at the connection between ε_{ij}^{*} and ε_{ij}^{0}, the *M* matrix written above is inverted which in turn leads Eq. (9) rewritten here for convenienc:

$$\begin{Bmatrix} \varepsilon_{11}^{*} \\ \varepsilon_{22}^{*} \\ \varepsilon_{33}^{*} \end{Bmatrix} = \frac{1}{A} \begin{bmatrix} A_1 & A_2 & A_3 \\ A_4 & A_5 & A_6 \\ A_7 & A_8 & A_9 \end{bmatrix} \begin{Bmatrix} \varepsilon_{11}^{0} \\ \varepsilon_{22}^{0} \\ \varepsilon_{33}^{0} \end{Bmatrix} \qquad \textbf{(A-6)}$$