MATLAB PROGRAM FOR THE NUMERICAL SOLUTION OF DUHAMEL CONVOLUTION INTEGRAL

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Abstract: *In the linear analysis of structures through modal superposition after decoupling the motion equations, there must be solved the motion equations for each degree of dynamic freedom, through numerical integration, by using the Duhamel convolution integral. To find a structural response to earthquake action it is also needed to determine the maximum spectral values in displacement, speed and acceleration for any accelerogram of the site; these values are determined by numerical integration with Duhamel integral In this paper it is presented a Matlab program for integrating numerical of Duhamel convolution integral with trapezoids method that can be used both for modal analysis and spectral analysis of structures.*

Key words: *Duhamel integral convolution, dynamic analysis by modal superposition, spectral analysis in earthquake engineering, dynamic response of the structures for random forces.*

1. Introduction

Dynamic analysis of civil structures is necessary when actions are time-varying. If dynamic analysis is linear, modal superposition principle can be used. To use modal superposition principle we must first achieve a modal decomposition and then make modal superposition. After decomposition modal vibration modes are obtained, which highlight modal deformed shapes associated with each mode of vibration and inertial mass corresponding to each mode of vibration.

Through modal decomposition two important results are obtained, one is qualitative and is represented by the form of own modes of vibration and another is quantitatively and indicates level of participation of the inertial masses in the movement of dynamic system.

The analysis of the vibration mode is very important because each vibration mode has a contribution for the motion of the dynamic system.

After spectral decomposition, modal superposition can be made, which involves composing significant vibration modes motion dynamic system.

In case of a system with n degrees of freedom (fig.1) dynamic equation of motion is:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\} \qquad (1)$$

where:

- $[M]$ inertial matrix of dynamic system;

- $[C]$ damping matrix of dynamic system;

- $[K]$ stiffness matrix of dynamic system;

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- $\{\ddot{x}\}$ accelerations vector for the n degrees of freedom dynamic;

- $\{\dot{x}\}$ velocity vector for the n degrees of freedom dynamic;

- $\{x\}$ displacement vector for the n degrees of freedom dynamic;

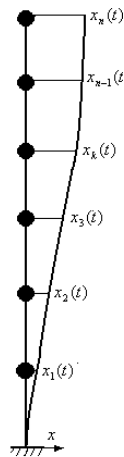- $\{F\}$ force vector for the n degrees of freedom dynamic.

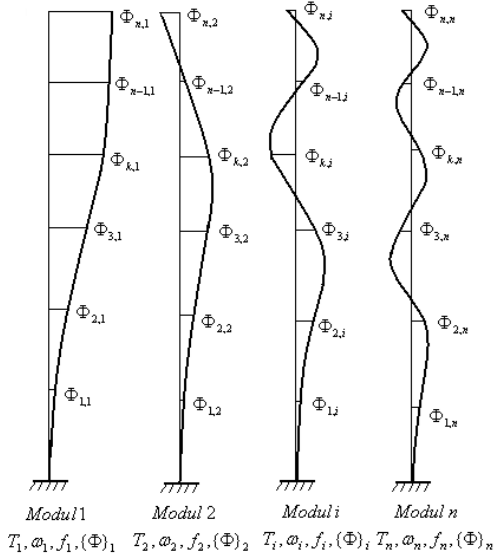Fig. 1. *The system with n degrees of freedom dynamic*

Fig. 2. *Vibrations mode and eigenvalues for a system with n degrees of freedom dynamic*

Instantaneous dynamic response on the degree of freedom k is obtained by linear superposition of instantaneous dynamic responses corresponding to the n modes of vibration fig.2:

$$x_k(t) = \sum_{i=1}^{n} x_{k,i}(t) \qquad (2)$$

To disconnect equations of motion the following linear transformation is used:

$$x_k(t) = \sum_{i=1}^{n} x_{k,i}(t) = \sum_{i=1}^{n} \Phi_{k,i} \eta_i(t) \qquad (3)$$

Relationship between natural coordinates dynamic system $x_k(t), k = 1...n$ and principal coordinates or modal $\eta_i(t), i = 1...n$ can be expressed by the modal matrix as follows:

$$\begin{Bmatrix} x_1 \\ x_2 \\ . \\ x_{n-1} \\ x_n \end{Bmatrix} = \begin{bmatrix} \Phi_{1,1} & \Phi_{1,2} & . & \Phi_{1,n-1} & \Phi_{1,n} \\ \Phi_{2,1} & \Phi_{2,2} & . & \Phi_{2,n-1} & \Phi_{2,n} \\ . & . & . & . & . \\ \Phi_{n-1,1} & \Phi_{n-1,2} & . & \Phi_{n-1,n-1} & \Phi_{n-1,n} \\ \Phi_{n,1} & \Phi_{n,2} & . & \Phi_{n,n-1} & \Phi_{n,n} \end{bmatrix} \begin{Bmatrix} \eta_1 \\ \eta_2 \\ . \\ \eta_{n-1} \\ \eta_n \end{Bmatrix}$$

Displacement vector velocity and acceleration on the dynamic degrees of freedom $k = 1...n$ become:

$$\{x(t)\} = [\Phi]\{\eta(t)\};$$

$$\{\dot{x}(t)\} = [\Phi]\{\dot{\eta}(t)\}; \qquad (4)$$

$$\{\ddot{x}(t)\} = [\Phi]\{\ddot{\eta}(t)\}.$$

where:

- $[\Phi]$ modal matrix

- $\{\eta(t)\}$ displacements vector in natural coordinates

- $\{\dot{\eta}(t)\}$ velocity vector in natural coordinates

- $\{\ddot{\eta}(t)\}$ accelerations vector in natural coordinates

Replacing in the system equations of motion (1) real coordinates with modal coordinates, the equations of motion is released.

Due to orthogonality of vectors associated vibration modes system of equations (1) is reduced to n decoupled equations of motion for each degree of freedom dynamic:

As the system of differential equations of motion is decoupled, we can write the corresponding equation of motion for each degree, as follows:

$$\{\ddot{\eta}_i(t)\} + 2\xi_i\omega_i\{\dot{\eta}_i(t)\} + \omega_i^2\{\eta(t)\} = \frac{F_i^*}{M_i^*} \quad (5)$$

$$i = 1...n$$

where:

- $F_i^* = \{\Phi\}_i^T\{F(t)\}$ corresponding equivalent force mode i;

- $M_i^* = \{\Phi\}_i^T[M]\{\Phi\}_i$ equivalent mass mode i;

- $\xi_i = \dfrac{\alpha + \beta\omega_i^2}{2\omega_i}$ corresponding damping mode i.

- $[C] = \alpha[M] + \beta[K]$ Rayleigh damping matrix

To solve decoupled equations of motion, Duhamel convolution integral can be used.

Displacement response obtained using The convolution integral Duhamel integral on each dynamic degrees of freedom is:

$$\eta_{ii}(t) = \frac{1}{M_i^*\omega_i^*}\int_0^t F^*(\tau)e^{-\xi_i\omega_i(t-\tau)}\sin\omega_i^*(t-\tau)d\tau$$

$$i = 1...n \qquad (6)$$

where,

- $\omega_i^* = \omega_i\sqrt{1-\xi^2}$ damped pulsation

If you know the answer in displacements on each dynamic degree of freedom then we can calculate velocities accelerations and inertial forces for dynamic system:

$$\{x(t)\} = [\Phi]\{\eta(t)\};$$

$$\{\dot{x}(t)\} = [\Phi]\{\dot{\eta}(t)\}; \qquad (7)$$

$$\{\ddot{x}(t)\} = [\Phi]\{\ddot{\eta}(t)\}.$$

The analysis of the dynamics of a system with n degrees of freedom dynamics can be seen as solving the system of equations is reduced to calculation convolution integral Duhamel.

For this reason this paper presents a Matlab program for numerical integration of convolution integral Duhamel.

If the $t = \tau$ apply an H impulse of the system, this is removed from equilibrium position with velocity $\dot{x} = H/m$, (fig.3) and performs free damped vibrations described by the equation :

$$x(t) = \frac{H}{m\omega^*}e^{-\beta_i(t-\tau)}\sin\omega^*(t-\tau) =$$

$$= \frac{H}{m\omega^*}e^{-\varsigma\omega^*(t-\tau)}\sin\omega_i^*(t-\tau) \quad for\, t > \tau \qquad (8)$$
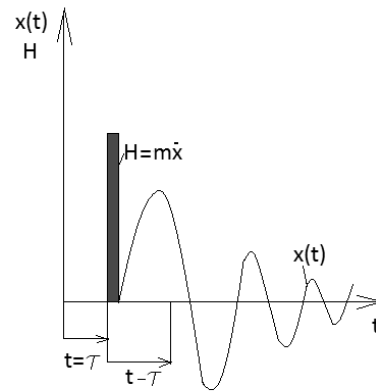

Fig. 3. *Free response in displacement due to an impulse finite H*

If the dynamic system random disturbing force acts $F(t) = F_0 f(t)$,(fig.4) equation

of motion of the dynamic system is:

$$\ddot{x}(t) + 2\beta \dot{x}(t) + \omega^2 x(t) = \frac{F(t)}{m}$$

Action of disturbing force $F(t)$ applied dynamic system at rest can be represented as a continuous elementary impulses $dH = F(\tau)d\tau$, on the time interval from $\tau = 0$ at $\tau = t$.

After applying elementary impulse $dH = F(\tau)d\tau$ displacement response of the dynamic system dynamic on degree of freedom is:

$$dx(t) = \frac{dH}{m\omega^*} e^{-\xi\omega(t-\tau)} \sin \omega^*(t-\tau)d\tau = $$

$$= \frac{F(\tau)}{m\omega^*} e^{-\varsigma\omega^*(t-\tau)} \sin \omega_i^*(t-\tau)d\tau \qquad (9)$$

Dynamic displacement response of the system is achieved by superposition of effects as a integral amount of motion generated by elementary pulse succession $dH = F(\tau)d\tau$ on the length of time that force acts [1]:

$$x(t) = \int_0^\tau \frac{F(\tau)}{m\omega^*} e^{-\varsigma\omega^*(t-\tau)} \sin \omega_i^*(t-\tau)d\tau = $$

$$= F_0 \int_0^\tau \frac{f(\tau)}{m\omega^*} e^{-\varsigma\omega^*(t-\tau)} \sin \omega_i^*(t-\tau)d\tau$$
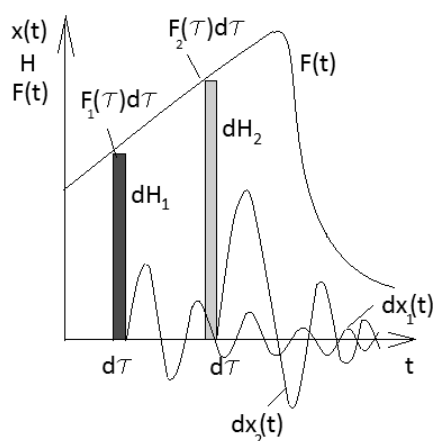

Fig. 4. *Displacement response obtained by superposition of the effects of infinitesimal impulses dH*

If disturbing force is applied to dynamic system while it is in motion due to initial displacement and velocity conditions, the displacement response is expressed:

$$x(t) = e^{-\beta t}[(x_0 \cos \omega t + \frac{\beta}{\omega} \sin \omega t) + \frac{\dot{x}_0}{\omega} \sin \omega t] +$$

$$+ F_0 \int_0^\tau \frac{f(\tau)}{m\omega^*} e^{-\varsigma\omega^*(t-\tau)} \sin \omega_i^*(t-\tau)d\tau$$

where,

- x_0 initial displacement for t=0

- $\dot{x}_0$ initial velocity for t=0

It can be seen that if the dynamic system with a single degree of freedom dynamic system to determine dynamic response Duhamel convolution integral must be calculated.

2. Mathematical aspects of numerical solution of the Duhamel integral

To integrate numerically convolution integral Duhamel, further is presented trapezoids method which is the simplest method used in practice.

Trapezoids method is based on approximating the area delimited by the graph of a function $f:[a,b] \to R$ with the area of a trapezoid. If the interval $[a,b]$ is divided into two subintervals $[a,x_1]$ and $[x_1,b]$ then integral of function f(x) can be approximated by the relation:

$$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + \int_{x_1}^b f(x)dx \approx$$

$$\approx \frac{h}{2}[f(a) + f(x_1)] + \frac{h}{2}[f(b) + f(x_1)] =$$

$$= \frac{h}{2}[f(a) + 2f(x_1) + f(b)] =$$

$$= \frac{b-a}{4}[f(a) + 2f(x_1) + f(b)]$$

To reduce the error of integration, the interval [a,b] may be divided in n subintervals with division $h = (b-a)/n$,

integral of function f(x) can be approximated by the relation:

$$\int_a^b f(x)dx = \int_a^{x_1} f(x)dx + ... + \int_{x_{n-11}}^b f(x)dx \approx$$

$$\approx \frac{h}{2}[f(a)+f(x_1)]+...\frac{h}{2}[f(b)+f(x_{n-1})]=$$

$$= \frac{h}{2}[f(a)+2f(x_1)+.....+2f(x_{n-1})+f(b)]=$$

$$= \frac{b-a}{2n}[f(a)+2f(x_1)+...+2f(x_{n-1})+f(b)]$$

Error in calculating introduced by the method of trapezoids depends on the integration step h and has the following expression on interval [xi xi+h]::

$$E_i(h) = \frac{h^3}{12} f''(\xi)$$

Full error is:

$$E_T(h) = \sum_{I=1}^{n-1} E_i(h) = \frac{h^3}{12} \sum_{I=1}^{n-1} f''(\xi)$$

3. Implementation of Matlab program for calculation Duhamel integral

For numerical calculation of the convolution integral Duhamel starts from response in displacement for a system with one degree of freedom:

$$x(t) = \int_0^\tau \frac{F(\tau)}{m\omega} e^{-\xi\omega(t-\tau)} \sin\omega(t-\tau)d\tau =$$

$$= \left(\int_0^\tau \frac{F(\tau)}{m\omega} e^{\xi\omega\tau} \cos\omega\tau \, d\tau\right) e^{-\xi\omega t} \sin\omega t -$$

$$- \left(\int_0^\tau \frac{F(\tau)}{m\omega} e^{\xi\omega\tau} \sin\omega\tau \, d\tau\right) e^{-\xi\omega t} \cos\omega t =$$

$$= A(t)e^{-\xi\omega t} \sin\omega t - B(t)e^{-\xi\omega t} \cos\omega t$$

where,

$$A(t) = \int_0^\tau \frac{F(\tau)}{m\omega} e^{\xi\omega\tau} \cos\omega\tau \, d\tau$$

$$B(t) = \int_0^\tau \frac{F(\tau)}{m\omega} e^{\xi\omega\tau} \sin\omega\tau \, d\tau$$

To solve the Duhamel integral trapezoids method will be used to calculate integrals A(t) and B(t) with Matlab program [2].

```
% Solving convolution integral Duhamel
%by numerical integration method of
%trapezoids
%Answer in diplacement , velocity and
%acceleration for dynamic sistem  one degree
%of freedom
% force ramp
    clear;clf;
    % Time step used
    h=0.2;
    % Mass system with 1GLD
    m=175;
    % Rigidity of the system with 1GLD
    k=7000;
    % Fraction of critical damping
    niu=5/100;
    % Pulsation system 1GLD without
%damping
    omega=sqrt(k/m);
    % Pulsation system 1GLD with
%damping
    omegad=omega*sqrt(1-niu^2);
    % Time interval that is computed
%response system with 1 DOF for
%displacement, velocity and acceleration
    tmax=10.4;
    % Force ramp function variation
    curbe=[50 0 2 10.4
        50 0 1  1 ];
    % Number of points of function curves
    npt=max(size(curbe));
    t=0;
  f0=interp1([curbe(1,2:npt)]',...
  [curbe(2,2:npt)]',t);
ya0=f0*1000*exp(niu*omega*t)*...
  cos(omegad*t)*(1000*m*omegad)^-1;
yb0=f0*1000*exp(niu*omega*t)*...
  sin(omegad*t)*(1000*m*omegad)^-1;
% Number of steps for numerical integration
  npas=tmax/h;
    subplot(2,1,1)
    for i=1:npas
      t=t+h;
      % Calculate the force at time t
      forta=interp1([curbe(1,2:npt)]',...
[curbe(2,2:npt)]',t)
      f1=forta*50;
```

```
        f(i)=f1;
        timp(i)=t;
% Variation representation force
 %function in time
        p=plot(t,f1,'--rs','LineWidth',2,...
            'MarkerEdgeColor','b',...
            'MarkerFaceColor','b',...
            'MarkerSize',2);
        hold on;
        grid on;
        % Calculation of values ya and yb
%from Duhamel convolution    integral
ya(i)=f1*1000*exp(niu*omega*t)*...
  cos(omegad*t)*(1000*m*omegad)^-1;
yb(i)=f1*1000*exp(niu*omega*t)*...
  sin(omegad*t)*(1000*m*omegad)^-1;
      end
    % Final values
    yaa=[ya0
      ya(1,:)' ];
    ybb=[yb0
      yb(1,:)'];
    % Method trapezoids
    t=0
    subplot(2,1,2)
     for i=2:npas+1
       t=t+h;
       A(1)=0;
       B(1)=0;
       A(i)=A(i-1)+h*(yaa(i)+yaa(i-1))/2;
       B(i)=B(i-1)+h*(ybb(i)+ybb(i-1))/2;
   % Response in motion for a ramp signal in
%case 1GLD dynamic system
   d(i)=(A(i)*exp(-niu*omega*t)*...
   sin(omegad*t)-...
   B(i)*exp(-niu*omega*t)*...
   cos(omegad*t))*(k/50);
% Graphical representation of the ratio of
%dynamic     displacement     and     static
%displacement
plot(t,d(i),'--rs','LineWidth',2,...
            'MarkerEdgeColor','r',...
            'MarkerFaceColor','r',...
            'MarkerSize',2);
        hold on;
        grid on;
      end
```

4. Results and conclusion

The program proposed in the paper has been tested in the case of a dynamical system with one degree of freedom dynamic.

Dynamic analysis was performed for two steps of integration h=0.2 and h=0.01.

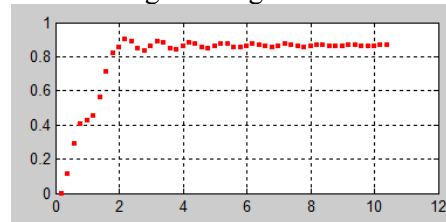The results after numerical integration are presented in fig.5 and fig.6

Fig. 5. *The response in displacement for a dynamic system with 1 degree of freedom, integration step 0,2*
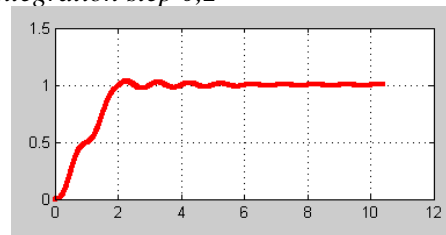

Fig. 6. *The response in displacement for a dynamic system with 1 degree of freedom, integration step 0,01*

The result obtained by numerical integration was compared with the exact solution, for the integration step h=0,2 obtain an error 13,7 % and if integration step is h=0.02 error was 0,03%.

Although the program presented is simple, it can be successfully used if the integration step is lower than where T/10, where T is own period for dynamic system

References

1. Ifrim, M.: *Dinamica structurilor si inginerie seismica.* Editura didactica si pedagogica, 1973.
2. Abdelwhab Kharab, Ronald B. Guenther,: *Introduction to Numerical Methods: A MATLAB Approach.* CRC Press Taylor &Francis Group, 2012.