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ON THE NUMERICAL SOLUTION OF NON-LINEAR ROTOR

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Abstract: *The present paper deals with the dynamical analysis and numerical solutions of non-linear rotor-bearing systems. The numerical solution is calculated using the Wilson-¯ method in conjunction with an iteration procedure. The model of rotor-bearing system comprises of a continuous elastic shaft mounted on several non-linear bearing. One or several disks are mounted on the shaft. Timoshenko beam model is adopted and gyroscopic effect is taken into account.*
Keywords: *non-linear rotor-bearings systems, numerical solutions*

1. INTRODUCTION

The behavior of dynamical systems undergoing time dependent changes (transients). Transient state analysis has been an active research area in many engineering problems.
For a linear mechanical system the equations of motion are

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\} \tag{1}$$

where $\{x\}$, $\{\dot{x}\}$, and $\{\ddot{x}\}$ refer to displacement, velocity, and acceleration vectors, respectively, and $[M]$, $[C]$, and $[K]$ are mass, damping and stiffness matrices.
In practical situation the systems are usually nonlinear and time-dependent. For example, the equation of motion for rotor-bearing systems with nonlinear bearings, systems that will be studied in this paper, is nonlinear

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K(\{x\})]^{nl}\{x\}=\{F\}\text{q} \tag{2}$$

The analytical solutions for such problems are difficult to obtain. The numerical methods have to be used.
Many numerical integration methods are available for approximate solutions of equations of motions. Direct numerical integrations methods is based on two ideas. First, instead of trying to satisfy dynamic governing equations at any time *t*, it is aimed to satisfy the equations only at discrete time interval Δt. The second idea is that, for each time interval Δt is assumed a specific type of variation of the displacement, $\{x\}$ velocity $\{\dot{x}\}$, and acceleration $\{\ddot{x}\}$.
Numerical integration methods are usually divided into two categories, implicit and explicit.
Consider the ODEs

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},t) \tag{3}$$

In an explicit numerical scheme, the ODEs are represented in terms of known values at a priori time step

$$\mathbf{x}_{i+1}=\mathbf{x}_i+\Delta t\ f(\mathbf{x}_i,t_i) \tag{4}$$

while in an implicit scheme

$$\mathbf{x}_{i+1}=\mathbf{x}_i+\Delta t\ f(\mathbf{x}_{i+1},t_i) \tag{5}$$

Explicit numerical schemes are conditionally stable. Implicit numerical schemes are unconditionally stable, i.e. do not impose a restriction on the size of time step Δt.
Thus several numerical integration schemes are available depending on the type of variations assumed for $\{x\}$, $\{\dot{x}\}$, and $\{\ddot{x}\}$ within each time interval Δt.
In this paper, the numerical time response solution for the non-linear rotor -bearing system is calculated using the Wilson-θ method in conjunction with an iteration procedure.

The Wilson-θ method is an extension of the linear-acceleration method. That is, within a time step the acceleration vector is proportional to time [1].

2. MODEL OF NON-LINEAR ROTOR-BEARING SYSTEMS

The model consists of a rotor treated as a continuous elastic shaft with several rigid disks, supported on the bearings with a non-linear behaviour. Consider that the dynamic equilibrium configuration of the rotor-bearing system the undeformed shaft is along the *x*- direction of an inertial *x, y, z* coordinate system. In the study of the lateral motion of the rotor, the displacement of any point is defined by two translations (v, w) and two rotations $\left(\ddot{\ }_y, \ddot{\ }_z\right)$. The model use the beam C^1 finite element type based on Timoshenko beam model. The beam finite element has two nodes. In the case of the dynamic analysis four degrees of freedom per node are considered: two displacements and two slopes measured in two perpendicular planes containing the beam, [4], [5].
It is well known that the behaviour of both lubricated journal bearings and rolling element bearings is strongly non-linear and can cause rotors to behave in a non-linear way. In this paper the nonlinearities are involved only in the "elastic" part of the system. The non-linear bearings have a cubic non-linear term [3], [6], where the force–displacement relation for a non-linear spring element can be written as a function of the complex displacement *z* by the law

$$f(z) = k\,(1+|z|^2)\,z \qquad (6)$$

This law is particularly well suited for modeling some rolling element bearings, in particular preloaded angular contact bearings [3], but has a more general application. Equation (6) has been widely used in non-linear dynamics, starting on the work of Duffing [2].
The global mass matrix and the damping matrix are the same as in the linearized model of the bearings, but the global stiffness matrix contains non-linear terms $k+\hat{k}u^2$ in nodal displacements, due to the stiffness matrix of the bearings

$$[k]^{nl} = \begin{bmatrix} \left[k_{yy}^{nl}\right] & | & [0] \\ --- & -- & --- \\ [0] & | & \left[k_{zz}^{nl}\right] \end{bmatrix} \qquad (7)$$

$$\left[k_{yy}^{nl}\right] = \begin{bmatrix} k_{yy}+\hat{k}\,v^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k_{yy} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{k}\,v^2 & 0 \\ 0 & 0 \end{bmatrix}$$
$$(8)$$
$$\left[k_{zz}^{nl}\right] = \begin{bmatrix} k_{zz}+\hat{k}\,w^2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} k_{zz} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{k}\,w^2 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, the global stiffness matrix can be written as the sum of two matrices by isolating the non-linear part

$$[K(\{x\})]^{nl} = [K] + \left[\hat{K}(\{x\})\right]^{nl} \qquad (9)$$

where the first matrix is the stiffness matrix of the structure which refers to the shaft and to the constant terms of the bearing stiffness and the second matrix appears due to the non-linearity of the bearings.
The equations of motions of anisotropic rotor-bearing systems which consist of a flexible non-uniform axisymmetric shaft, rigid disk and anisotropic bearings are obtained in second order form, by assembling the element matrices and may be written as

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K(\{x\})]^{nl}\{x\} = \{F\} \qquad (10)$$

3. NUMERICAL SOLUTION. THE WILSON – Θ METHOD

The numerical time response solution for the non-linear system is calculated using the Wilson-θ method in conjunction with an iteration procedure. From the Wilson-θ method the resulting equations for acceleration and velocity vector can be expressed as

$$\{\ddot{x}\}_{t+{}^{-}\Delta t} = \frac{6}{{}^{-2}\Delta t^2}\left(\{x\}_{t+{}^{-}\Delta t} - \{x\}_t\right) - \frac{6}{{}^{-}\Delta t}\{\dot{x}\}_t - 2\{\ddot{x}\}_t \qquad (11)$$

$$\{\dot{x}\}_{t+{}^{-}\Delta t} = \frac{3}{{}^{-}\Delta t}\left(\{x\}_{t+{}^{-}\Delta t} - \{x\}_t\right) - 2\{\dot{x}\}_t - \frac{{}^{-}\Delta t}{2}\{\ddot{x}\}_t \qquad (12)$$

From the equation of motion (10) written at the time $t+\theta\Delta t$ by using the equations (11) and (12) we obtain the non-linear algebraic system

$$\left[\tilde{K}\right]^{nl}\{x\}_{t+\theta\Delta t}=\left\{\tilde{F}\right\} \tag{13}$$

with $\{x\}_{t+\theta\Delta t}$ as the unknown. In the equation (13) we used the following notations

$$\left[\tilde{K}\right]=[K]+\frac{6}{(\theta\Delta t)^2}[M]+\frac{3}{\theta\Delta t}[C] \tag{14}$$

$$\begin{aligned}\left\{\tilde{F}\right\}=\{F\}_t+\theta\left(\{F\}_{t+\Delta t}-\{F\}_t\right)+[M]\left(\frac{6}{(\theta\Delta t)^2}\{x\}_t+\frac{6}{\theta\Delta t}\{\dot{x}\}_t+2\{\ddot{x}\}_t\right)+\\ +[C]\left(\frac{3}{\theta\Delta t}\{x\}_t+2\{\dot{x}\}_t+\frac{\theta\Delta t}{2}\{\ddot{x}\}_t\right)\end{aligned} \tag{15}$$

In the non-linear rotors case, the stiffness matrix $\left[\tilde{K}\right]$ in Eq. (13) has non-linear terms, which depend on the values of the elements in vector $\{x\}_{t+\theta\Delta t}$. Equation (13) is a set of non-linear algebraic equations now. Therefore an iteration procedure is utilized in conjunction with the Wilson-θ method to find $\{x\}_{t+\theta\Delta t}$ and then the displacements $\{x\}_{t+\Delta t}$. The following steps describe the numerical procedure:

1. At $t=0$ specify initial conditions $\{x\}_0,\{\dot{x}\}_0$.
2. From the Eq. (13), using the known initial conditions from step 1, $\{\ddot{x}\}_0$ is calculated as

$$\{\ddot{x}\}_0=[M]^{-1}\left(\{F\}_{t=0}-[C]\{\dot{x}\}_0-[K]^{nl}\{x\}_0\right). \tag{16}$$

3. $i=i+1$
4. For the time step *i* at the moment of time *t*, assume a displacement vector $\{x\}_t\equiv\{x\}_i=\{x\}_i^*$.
5. Calculate the non-linear terms of the stiffness matrix $\left[\tilde{K}\right]$ using values from the assumed displacement vector.
6. Calculate displacement vector $\{x\}_{t+\Delta t}\equiv\{x\}_{i+1}$ using

$$\{x\}_{t+\Delta t}=\{x\}_t+\Delta t\{\dot{x}\}_t+\frac{\Delta t^2}{6}\left(\{\ddot{x}\}_{t+\Delta t}+2\{\ddot{x}\}_t\right) \tag{17}$$

and the stiffness matrix from Step (5).

7. The vector $\{x\}_i^* := \left(\{x\}_i^*+\{x\}_{i+1}\right)/2$ is modified.
8. Compare $\{x\}_{i+1}$ with assumed displacement vector

$$\sqrt{\left(\{x\}_{i+1}-\{x\}_i^*\right)^T\left(\{x\}_{i+1}-\{x\}_i^*\right)}<tol \tag{18}$$

9. If the difference is not within specified tolerances (tol) use an average value of the assumed displacement vector from Step (7) and the calculated displacement vector from Step (6) for the new assumed vector and return to Step (4). If the difference is within tolerance then update the assumed vector for t_{i+1} and go to Step (3). Continue for $i=N$ iterations to obtain the steady state solution.

Numerical example.

The model comprises of a continuous elastic shaft mounted on tree nonlinear bearing, Figure 1. One disk is mounted on the shaft. Timoshenko beam model is adopted and gyroscopic effect is taken into account [4].

Table 1: Rotor data – numerical example

Shaft :	Disk :	Bearings:
L = 1.2 m	M = 75 Kg	k_{yy} = 5 x 10^8 N/m
a = 0.3 m	J_T = .190 Kg m^2	k_{zz} = 3 x 10^8 N/m
$d_1 = d_2 = d_3$ = 0 .08 m	J_P = 0.368 Kg m^2	$k_{yz} = k_{zy}$ =0
E = 2.068e11 N/m^2	e = 0.01 m	$c_{yy} = c_{zz}$ =1 x 10^4 Ns/m
ρ = 7.833 Kg/m^3		$c_{zy} = c_{yz}$ = 0

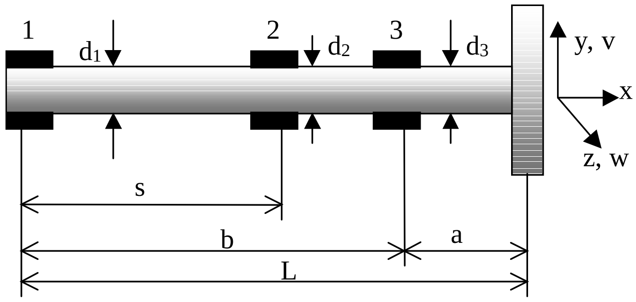

Figure 1: Rotor configuration

where: J_T, J_P are transverse, respectively, polar mass moment of inertia; k_{ij}, c_{ij} (i, j = y, z) are stiffness and damping coefficients; *e* is the eccentricity of the disk. In this numerical example in the equations (7) and (8) we deal with the values: $\hat{k} = 10^{14}\ N/m^3$, *s* = 0.25 m, *nrot* = 60, *nstep* = 4.096; N/m.

In an implicit schema the difference equations are combined with the equation of motion and the displacement are calculated directly by solving the equations. The graph response, vertical displacements of the flywheel node is shown in Figure 2.

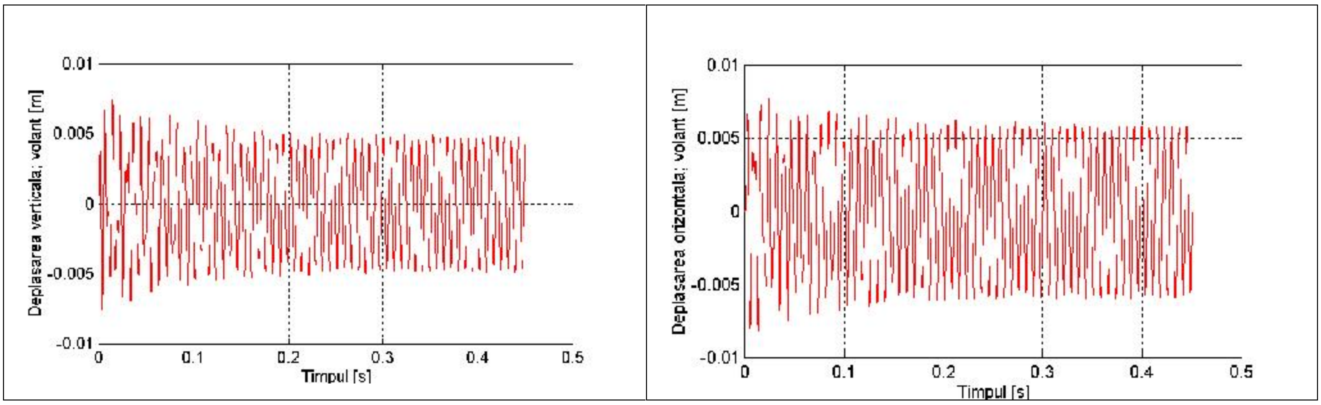

Figure 2: Vertical movement of the flywheel node

4. CONCLUSION

There is a basic approach to numerically evaluate the dynamic response of non-linear rotors. The implicit scheme Wilson-Theta method in conjunction with an iteration procedure is useful to a multipledegrees-of-freedom (MDOF) non-linear system with non-proportional damping.

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