6th International Conference
ðComputational Mechanics and Virtual Engineering ð
COMET 2015
15-16 October 2015, Braşov, Romania

DYNAMICS OF A SYSTEM OF RIGID BODIES WITH GENERAL CONSTRAINTS BY A MULTIBODY APPROACH

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Abstract: *This paper generalizes the equations of motion for a single rigid body, published in a previous work, for the case of a system containing an arbitrary number of rigid bodies. It is not important the way in which the rigid bodies are linked one to another. We also present an application to highlight the theory.*
Keywords): *constraint, multibody, equations of motion, reactions*

1. INTRODUCTION

The study of the multibody systems is a great task of nowadays researchers. The general case of the rigid body with arbitrary constraints is published in [1]. In this paper we will generalize the equation of motion published there for the case of a mechanical system with an arbitrary number of rigid bodies linked one to another.
The matrix equation of motion may be obtained in two ways. The first approach is to use the general theorems (the theorem of momentum and the theorem of moment of momentum). In this approach we make the following notations

$$[\mathbf{M_q}] = \begin{bmatrix} [\mathbf{m}] & [\mathbf{A}][\mathbf{S}]^{\mathrm{T}}[\mathbf{Q}] \\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{S}][\mathbf{A}]^{\mathrm{T}} & [\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_O][\mathbf{Q}] \end{bmatrix},\quad \{\mathbf{F_q}\} = \begin{bmatrix} \{\mathbf{F_s}\} \\ \{\mathbf{F_\beta}\} \end{bmatrix},\quad \{\tilde{\mathbf{F}}_\mathbf{q}\} = \begin{bmatrix} \{\tilde{\mathbf{F}}_\mathbf{s}\} \\ \{\tilde{\mathbf{F}}_\boldsymbol{\beta}\} \end{bmatrix},\quad \{\tilde{\mathbf{F}}_\mathbf{s}\} = -\left[[\mathbf{A}][\mathbf{S}]^{\mathrm{T}}[\dot{\mathbf{Q}}] + [\dot{\mathbf{A}}][\mathbf{S}]^{\mathrm{T}}[\mathbf{Q}]\right]\{\dot{\boldsymbol{\beta}}\},$$

$$\{\tilde{\mathbf{F}}_\boldsymbol{\beta}\} = -\left[[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_O][\dot{\mathbf{Q}}] + [\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\omega}][\mathbf{J}_O][\mathbf{Q}]\right]\{\dot{\boldsymbol{\beta}}\}, \tag{1}$$

and the matrix equation of motion takes the form

$$[\mathbf{M_q}]\{\ddot{\mathbf{q}}\} = \{\mathbf{F_q}\} + \{\tilde{\mathbf{F}}_\mathbf{q}\} \tag{2}$$

where

$$[\mathbf{A}] = [\boldsymbol{\psi}][\boldsymbol{\theta}][\boldsymbol{\varphi}],\ [\mathbf{A}_\psi] = [\mathbf{U}_\psi][\mathbf{A}],\ [\mathbf{A}_\theta] = [\boldsymbol{\psi}][\mathbf{U}_\theta][\boldsymbol{\varphi}]^{\mathrm{T}}[\mathbf{A}],\ [\mathbf{A}_\varphi] = [\mathbf{A}][\mathbf{U}_\varphi],\ [\dot{\mathbf{A}}] = \dot{\psi}[\mathbf{A}_\psi] + \dot{\theta}[\mathbf{A}_\theta] + \dot{\varphi}[\mathbf{A}_\varphi],$$

$$[\boldsymbol{\omega}] = [\mathbf{A}]^{\mathrm{T}}[\dot{\mathbf{A}}],\ [\boldsymbol{\theta}_Q] = \left[[\boldsymbol{\theta}]^{\mathrm{T}}\{\mathbf{u}_\psi\}\ \{\mathbf{u}_\theta\}\ \{\mathbf{u}_\varphi\}\right],\ [\mathbf{Q}] = [\boldsymbol{\varphi}]^{\mathrm{T}}[\boldsymbol{\theta}_Q],\ \{\boldsymbol{\omega}\} = [\mathbf{Q}]\begin{bmatrix}\dot{\psi}\\ \dot{\theta}\\ \dot{\varphi}\end{bmatrix},\ [\mathbf{S}] = \begin{bmatrix} 0 & -mz_C & my_C \\ mz_C & 0 & -mx_C \\ -my_C & mx_C & 0 \end{bmatrix},$$

$$[\dot{\mathbf{Q}}] = -\dot{\varphi}\left[[\mathbf{U}_\varphi][\boldsymbol{\varphi}]^{\mathrm{T}}\{\mathbf{u}_\psi\}\ [\mathbf{U}_\varphi][\boldsymbol{\varphi}]^{\mathrm{T}}\{\mathbf{u}_\theta\}\ \{\mathbf{0}\}\right] - \dot{\theta}\left[[\boldsymbol{\varphi}]^{\mathrm{T}}[\boldsymbol{\theta}]^{\mathrm{T}}[\mathbf{U}_\theta]\{\mathbf{u}_\psi\}\ \{\mathbf{0}\}\ \{\mathbf{0}\}\right]. \tag{3}$$

If we use the Lagrange equations, then we denote by q_k, and F_{q_k}, $k = \overline{1,n}$, the generalized coordinates, and forces, respectively, and the Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E_c}{\partial \dot{q}_k}\right) - \frac{\partial E_c}{\partial q_k} = F_{q_k},\ k = \overline{1,n}, \tag{4}$$

where E_c is the kinetic energy, and the generalized forces contain both the given and the constraint forces.
Using the matrix notations

$$\left\{\frac{\mathrm{d}E_c}{\partial \dot{\mathbf{q}}}\right\} \equiv \left[\frac{\partial E_c}{\partial \dot{q}_1}\ \frac{\partial E_c}{\partial \dot{q}_2}\ \cdots\ \frac{\partial E_c}{\partial \dot{q}_n}\right]^{\mathrm{T}},\ \left\{\frac{\mathrm{d}E_c}{\partial \mathbf{q}}\right\} \equiv \left[\frac{\partial E_c}{\partial q_1}\ \frac{\partial E_c}{\partial q_2}\ \cdots\ \frac{\partial E_c}{\partial q_n}\right]^{\mathrm{T}},\ \{\mathbf{F_q}\} \equiv \left[F_{q_1}\ F_{q_2}\ \cdots\ F_{q_n}\right]^{\mathrm{T}}, \tag{5}$$

one obtains the matrix form of the Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial E_c}{\partial \dot{\mathbf{q}}}\right\} - \left\{\frac{\partial E_c}{\partial \mathbf{q}}\right\} = \{\mathbf{F_q}\}. \tag{6}$$

Denoting

$$\{\mathbf{s}\} = [X_O\ Y_O\ Z_O]^{\mathrm{T}},\ \{\boldsymbol{\beta}\} = [\psi\ \theta\ \varphi]^{\mathrm{T}},\ \{\mathbf{q}\} = [X_O\ Y_O\ Z_O\ \psi\ \theta\ \varphi]^{\mathrm{T}}, \tag{7}$$

the kinetic energy reads

$$E_c = \frac{1}{2}\{\dot{\mathbf{q}}\}^{\mathrm{T}}[\mathbf{M_q}]\{\dot{\mathbf{q}}\} \tag{8}$$

wherefrom we get

$$\left\{\frac{\partial E_c}{\partial \dot{\mathbf{q}}}\right\} = [\mathbf{M_q}]\{\dot{\mathbf{q}}\},\ \left\{\frac{\partial E_c}{\partial \mathbf{q}}\right\} = \frac{1}{2}\left[\{\dot{\mathbf{q}}\}^{\mathrm{T}}\frac{\partial[\mathbf{M_q}]}{\partial X_O}\{\dot{\mathbf{q}}\}\ \dots\ \{\dot{\mathbf{q}}\}^{\mathrm{T}}\frac{\partial[\mathbf{M_q}]}{\partial \varphi}\{\dot{\mathbf{q}}\}\right]^{\mathrm{T}} \tag{9}$$

and the equation (4) becomes

$$[\mathbf{M_q}]\{\ddot{\mathbf{q}}\} + [\dot{\mathbf{M}}_\mathbf{q}]\{\dot{\mathbf{q}}\} - \left\{\frac{\partial E_c}{\partial \mathbf{q}}\right\} = \{\mathbf{F_q}\}. \tag{10}$$

Denoting

$$\left\{\tilde{\tilde{\mathbf{F}}}_\mathbf{q}\right\} = -[\dot{\mathbf{M}}_\mathbf{q}]\{\dot{\mathbf{q}}\} + \left\{\frac{\partial E_c}{\partial \mathbf{q}}\right\}, \tag{11}$$

it results

$$[\mathbf{M_q}]\{\ddot{\mathbf{q}}\} = \{\mathbf{F_q}\} + \left\{\tilde{\tilde{\mathbf{F}}}_\mathbf{q}\right\}. \tag{12}$$

2. THE MATRIX EQUATION OF MOTION FOR A RIGID BODY WITH CONSTRAINTS

One may prove [1] that the equations (2) and (12) are equivalent, that is, the matrices $\{\tilde{\mathbf{F}}_\mathbf{q}\}$, $\left\{\tilde{\tilde{\mathbf{F}}}_\mathbf{q}\right\}$ are one and the same. In this way, the generalized and the constraint forces can be replace by the sum between the matrix $\{\mathbf{F_q}\}$ of the generalized given forces and the matrix $[\mathbf{B}]^{\mathrm{T}}\{\boldsymbol{\lambda}\}$, where $[\mathbf{B}]$ is the matrix of constraints, and $\{\boldsymbol{\lambda}\}$ is the matrix of the Lagrange multipliers. It results the differential equation of motion

$$\begin{bmatrix} [\mathbf{M}] & -[\mathbf{B}]^{\mathrm{T}} \\ [\mathbf{B}] & [\mathbf{0}] \end{bmatrix}\begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \{\boldsymbol{\lambda}\} \end{bmatrix} = \begin{bmatrix} \{\mathbf{F_q}\} + \{\tilde{\mathbf{F}}_\mathbf{q}\} \\ \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}, \tag{13}$$

where

$[\mathbf{B}]\{\dot{\mathbf{q}}\} = \{\mathbf{C}\}$ (14)

is the equation of the constraints, and

$$\{\mathbf{s}\} = [X_O\ Y_O\ Z_O]^{\mathrm{T}},\quad \{\boldsymbol{\beta}\} = [\psi\ \theta\ \varphi]^{\mathrm{T}},\quad \{\mathbf{q}\} = [X_O\ Y_O\ Z_O\ \psi\ \theta\ \varphi]^{\mathrm{T}},\quad [\mathbf{M}] = \begin{bmatrix} [\mathbf{m}] & [\mathbf{A}][\mathbf{S}]^{\mathrm{T}}[\mathbf{Q}] \\ [\mathbf{Q}]^{\mathrm{T}}[\mathbf{S}][\mathbf{A}]^{\mathrm{T}} & [\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_O][\mathbf{Q}] \end{bmatrix},$$

$$\{\tilde{\mathbf{F}}_\mathbf{q}\} = \begin{bmatrix} \{\tilde{\mathbf{F}}_\mathbf{s}\} \\ \{\tilde{\mathbf{F}}_\boldsymbol{\beta}\} \end{bmatrix},\ \{\tilde{\mathbf{F}}_\mathbf{s}\} = -\left[[\mathbf{A}][\mathbf{S}]^{\mathrm{T}}[\dot{\mathbf{Q}}] + [\dot{\mathbf{A}}][\mathbf{S}]^{\mathrm{T}}[\mathbf{Q}]\right]\{\dot{\boldsymbol{\beta}}\},\ \{\tilde{\mathbf{F}}_\boldsymbol{\beta}\} = -\left[[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_O][\dot{\mathbf{Q}}] + [\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\omega}][\mathbf{J}_O][\mathbf{Q}]\right]\{\dot{\boldsymbol{\beta}}\}, \tag{14}$$

The matrix of constraints $[\mathbf{B}]$ has the general form

$$[\mathbf{B}] = \begin{bmatrix} B_{11} & B_{12} & \dots & B_{16} \\ B_{21} & B_{22} & \dots & B_{26} \\ \dots & \dots & \dots & \dots \\ B_{p1} & B_{p2} & \dots & B_{p6} \end{bmatrix}; \tag{15}$$

hence, the generalized force that corresponds to the constraint of index i will have the components

$$\{\mathbf{F}_{G\mathbf{s}}^{(i)}\} = \lambda_i\begin{bmatrix} B_{i1} \\ B_{i2} \\ B_{i3} \end{bmatrix},\ \{\mathbf{F}_{G\boldsymbol{\beta}}^{(i)}\} = \lambda_i\begin{bmatrix} B_{i4} \\ B_{i5} \\ B_{i6} \end{bmatrix}, \tag{16}$$

in which $\lambda_i B_{i1}$, $\lambda_i \mathrm{B}_{i2}$ and $\lambda_i B_{i3}$ represent the projections of the force of constraint onto the axes of the fixed reference frame, while the projections of the moments about the point O, onto the mobile axes are obtained from the expression $\left[[\mathbf{Q}]^{-1}\right]^{\mathrm{T}}\left\{\mathbf{F}_{G\boldsymbol{\beta}}^{(i)}\right\}$.

3. THE MOVING EQUATION FOR A SYSTEM OF RIGID BODIES

The moving equation (13) may be easily generalized for the case of n rigid bodies with constraints. We make the following modifications:

– the matrix of constraints $[\mathbf{B}]$ reads

$$[\mathbf{B}]^{\mathrm{T}} = \begin{bmatrix} \frac{\partial f_1}{\partial X_{O_1}} & \cdots & \frac{\partial f_i}{\partial X_{O_1}} & \cdots & \frac{\partial f_p}{\partial X_{O_1}} \\ \frac{\partial f_1}{\partial Y_{O_1}} & \cdots & \frac{\partial f_i}{\partial Y_{O_1}} & \cdots & \frac{\partial f_p}{\partial Y_{O_1}} \\ \frac{\partial f_1}{\partial Z_{O_1}} & \cdots & \frac{\partial f_i}{\partial Z_{O_1}} & \cdots & \frac{\partial f_p}{\partial Z_{O_1}} \\ \frac{\partial f_1}{\partial \psi_1} & \cdots & \frac{\partial f_i}{\partial \psi_1} & \cdots & \frac{\partial f_p}{\partial \psi_1} \\ \frac{\partial f_1}{\partial \theta_1} & \cdots & \frac{\partial f_i}{\partial \theta_1} & \cdots & \frac{\partial f_p}{\partial \theta_1} \\ \frac{\partial f_1}{\partial \varphi_1} & \cdots & \frac{\partial f_i}{\partial \varphi_1} & \cdots & \frac{\partial f_p}{\partial \varphi_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_1}{\partial X_{O_n}} & \cdots & \frac{\partial f_i}{\partial X_{O_n}} & \cdots & \frac{\partial f_p}{\partial X_{O_n}} \\ \frac{\partial f_1}{\partial Y_{O_n}} & \cdots & \frac{\partial f_i}{\partial Y_{O_n}} & \cdots & \frac{\partial f_p}{\partial Y_{O_n}} \\ \frac{\partial f_1}{\partial Z_{On}} & \cdots & \frac{\partial f_i}{\partial Z_{On}} & \cdots & \frac{\partial f_p}{\partial Z_{On}} \\ \frac{\partial f_1}{\partial \psi_n} & \cdots & \frac{\partial f_i}{\partial \psi_n} & \cdots & \frac{\partial f_p}{\partial \psi_n} \\ \frac{\partial f_1}{\partial \theta_n} & \cdots & \frac{\partial f_i}{\partial \theta_n} & \cdots & \frac{\partial f_p}{\partial \theta_n} \\ \frac{\partial f_1}{\partial \varphi_n} & \cdots & \frac{\partial f_i}{\partial \varphi_n} & \cdots & \frac{\partial f_p}{\partial \varphi_n} \end{bmatrix}, \tag{17}$$

in which $f_i(\mathbf{q}) = 0$, $i = \overline{1, p}$, are the constraints;

– the matrix $[\mathbf{M}]$ has the expression

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{M}_1] & [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{M}_2] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{M}_3] & \cdots & [\mathbf{0}] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] & \cdots & [\mathbf{M}_n] \end{bmatrix}, \tag{18}$$

where

$$[\mathbf{M}_k] = \begin{bmatrix} [m_k] & [\mathbf{A}_k][\mathbf{S}_k]^{\mathrm{T}}[\mathbf{Q}_k] \\ [\mathbf{Q}_k]^{\mathrm{T}}[\mathbf{S}_k][\mathbf{A}_k]^{\mathrm{T}} & [\mathbf{Q}_k]^{\mathrm{T}}[\mathbf{J}_{O_k}][\mathbf{Q}_k] \end{bmatrix}, \tag{19}$$

O_k being the origin of the mobile reference system linked to the rigid body k, $k = \overline{1, n}$;

– the matrices $\{\mathbf{s}\}$ and $\{\boldsymbol{\beta}\}$ take the form

$$\{\mathbf{s}\} = \begin{bmatrix} X_{O_1} & Y_{O_1} & Z_{O_1} & \cdots & X_{O_n} & Y_{O_n} & Z_{O_n} \end{bmatrix}^{\mathrm{T}},\ \{\boldsymbol{\beta}\} = \begin{bmatrix} \psi_1 & \theta_1 & \varphi_1 & \cdots & \psi_n & \theta_n & \varphi_n \end{bmatrix}^{\mathrm{T}}, \tag{20}$$

that is,

$$\{\mathbf{s}\} = \left[\{\mathbf{s}_1\}^{\mathrm{T}}\ \{\mathbf{s}_2\}^{\mathrm{T}}\ \cdots\ \{\mathbf{s}_n\}^{\mathrm{T}}\right]^{\mathrm{T}},\ \{\boldsymbol{\beta}\} = \left[\{\boldsymbol{\beta}_1\}^{\mathrm{T}}\ \{\boldsymbol{\beta}_2\}^{\mathrm{T}}\ \cdots\ \{\boldsymbol{\beta}_n\}^{\mathrm{T}}\right]^{\mathrm{T}}; \tag{21}$$

– the vector $\{\mathbf{q}\}$ writes as

$$\{\mathbf{q}\} = \left[\{\mathbf{s}_1\}^{\mathrm{T}}\ \ \{\boldsymbol{\beta}_1\}^{\mathrm{T}}\ \ \ldots\ \ \{\mathbf{s}_n\}^{\mathrm{T}}\ \ \{\boldsymbol{\beta}_n\}^{\mathrm{T}}\right]^{\mathrm{T}} \tag{22}$$

and it has $6n$ components;

– the vector $\{\tilde{\mathbf{F}}_{\mathbf{q}}\}$ becomes

$$\{\tilde{\mathbf{F}}_{\mathbf{q}}\} = \left[\{\tilde{\mathbf{F}}_{\mathbf{s}_1}\}^{\mathrm{T}}\ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_1}\}^{\mathrm{T}}\ \ldots\ \{\tilde{\mathbf{F}}_{\mathbf{s}_n}\}^{\mathrm{T}}\ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_n}\}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{23}$$

where $\{\tilde{\mathbf{F}}_{\mathbf{s}_k}\}$ and $\{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_k}\}$, $k = \overline{1, n}$, have the same known expressions;

Let us remark that if there is no link between the rigid bodies, then the matrix of constraints takes the form

$$[\mathbf{B}] = \begin{bmatrix} \left[\dfrac{\partial \mathbf{f}_1}{\partial \mathbf{q}_1}\right] & [\mathbf{0}] & \cdots & [\mathbf{0}] \\ [\mathbf{0}] & \left[\dfrac{\partial \mathbf{f}_2}{\partial \mathbf{q}_2}\right] & \cdots & [\mathbf{0}] \\ \cdots & \cdots & \cdots & \cdots \\ [\mathbf{0}] & [\mathbf{0}] & \cdots & \left[\dfrac{\partial \mathbf{f}_n}{\partial \mathbf{q}_n}\right] \end{bmatrix}, \tag{24}$$

where $\mathbf{f}_k = \left[f_{p_1}\ \ f_{p_2}\ \cdots\ f_{p_k}\right]^{\mathrm{T}}$ are the constraints of the rigid solid k, $k = \overline{1, n}$, each constraint of this kind containing only the parameters $\{\mathbf{q}_k\}$. In this way, one obtains n independent equations of type (13).

4. EXAMPLE

We consider two bars AC, and CB of lengths l_1, and l_2, masses m_1, and m_2, respectively, linked one to another by a spherical joint at the point C. The mobile reference systems $O_1x_1y_1z_1$, $O_2x_2y_2z_2$ are principal central system of inertia for which one knows the values J_{x_1}, J_{y_1}, J_{z_1}, J_{x_2}, J_{y_2}, J_{z_2}. For the fixed reference system O_0XYZ the axis O_0Z is vertical ascendant.

Choosing as rotational parameters the Bryan angles for each body, we have

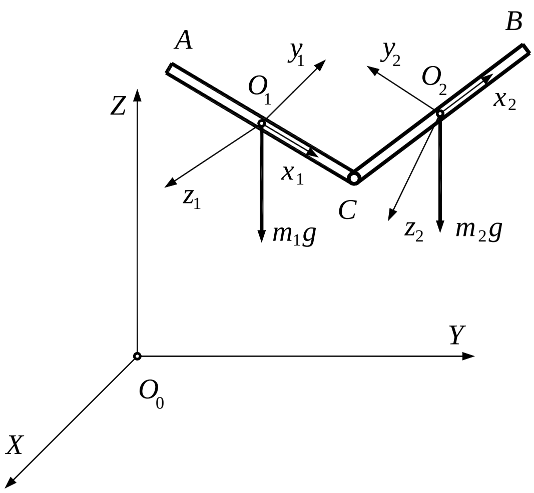

Figure 1: Example

$$[\boldsymbol{\psi}_i] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi_i & -\sin\psi_i \\ 0 & \sin\psi_i & \cos\psi_i \end{bmatrix},\ [\boldsymbol{\theta}_i] = \begin{bmatrix} \cos\theta_i & 0 & \sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 0 & \cos\theta_i \end{bmatrix},\ [\boldsymbol{\varphi}_i] = \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

$$\left[\mathbf{A}_i\right] = \left[\boldsymbol{\psi}_i\right]\left[\boldsymbol{\theta}_i\right]\left[\boldsymbol{\varphi}_i\right] = \begin{bmatrix} \mathrm{c}\theta_i\mathrm{c}\varphi_i & -\mathrm{c}\theta_i\mathrm{s}\varphi_i & \mathrm{s}\theta_i \\ \mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i + \mathrm{c}\psi_i\mathrm{s}\varphi_i & -\mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i + \mathrm{c}\psi_i\mathrm{c}\varphi_i & -\mathrm{s}\psi_i\mathrm{c}\theta_i \\ \mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i + \mathrm{s}\psi_i\mathrm{s}\varphi_i & \mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i + \mathrm{s}\psi_i\mathrm{c}\varphi_i & \mathrm{c}\psi_i\mathrm{c}\theta_i \end{bmatrix}, \tag{26}$$

$$\left[\mathbf{S}_i\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \left[\mathbf{U}_{\psi_i}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix},\ \left[\mathbf{U}_{\theta_i}\right] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix},\ \left[\mathbf{U}_{\varphi_i}\right] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{27}$$

$$\left[\boldsymbol{\psi}_{p_i}\right] = \left[\mathbf{U}_{\psi_i}\right]\left[\boldsymbol{\psi}_i\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\psi_i & \cos\psi_i \\ 0 & \cos\psi_i & -\sin\psi_i \end{bmatrix}, \qquad \left[\boldsymbol{\theta}_{p_i}\right] = \left[\mathbf{U}_{\theta_i}\right]\left[\boldsymbol{\theta}_i\right] = \begin{bmatrix} -\sin\theta_i & 0 & \cos\theta_i \\ 0 & 0 & 0 \\ \cos\theta_i & 0 & -\sin\theta_i \end{bmatrix},$$

$$\left[\boldsymbol{\varphi}_{p_i}\right] = \left[\mathbf{U}_{\varphi_i}\right]\left[\boldsymbol{\varphi}_i\right] = \begin{bmatrix} -\sin\varphi_i & -\cos\varphi_i & 0 \\ \cos\varphi_i & -\sin\varphi_i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{28}$$

$$\left[\mathbf{A}_{\psi_i}\right] = \left[\boldsymbol{\psi}_{p_i}\right]\left[\boldsymbol{\theta}_i\right]\left[\boldsymbol{\varphi}_i\right] = \begin{bmatrix} 0 & 0 & 0 \\ \mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i - \mathrm{s}\psi_i\mathrm{s}\varphi_i & -\mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i - \mathrm{s}\psi_i\mathrm{c}\varphi_i & -\mathrm{c}\psi_i\mathrm{c}\theta_i \\ \mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i + \mathrm{c}\psi_i\mathrm{s}\varphi_i & -\mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i + \mathrm{c}\psi_i\mathrm{c}\varphi_i & -\mathrm{s}\psi_i\mathrm{c}\theta_i \end{bmatrix},$$

$$\left[\mathbf{A}_{\theta i}\right] = \left[\boldsymbol{\psi}_i\right]\left[\boldsymbol{\theta}_{p_i}\right]\left[\boldsymbol{\varphi}_i\right] = \begin{bmatrix} -\mathrm{s}\theta_i\mathrm{c}\varphi_i & \mathrm{s}\theta_i\mathrm{s}\varphi_i & \mathrm{c}\theta_i \\ \mathrm{s}\psi_i\mathrm{c}\theta_i\mathrm{c}\varphi_i & -\mathrm{s}\psi_i\mathrm{c}\theta_i\mathrm{s}\varphi_i & \mathrm{s}\psi_i\mathrm{s}\theta_i \\ -\mathrm{c}\varphi_i\mathrm{c}\theta_i\mathrm{c}\varphi_i & \mathrm{s}\psi_i\mathrm{c}\theta_i\mathrm{s}\varphi_i & -\mathrm{c}\psi_i\mathrm{s}\theta_i \end{bmatrix},$$

$$\left[\mathbf{A}_{\varphi_i}\right] = \left[\boldsymbol{\psi}_i\right]\left[\boldsymbol{\theta}_i\right]\left[\boldsymbol{\varphi}_{p_i}\right] = \begin{bmatrix} -\mathrm{c}\theta_i\mathrm{s}\varphi_i & -\mathrm{c}\theta_i\mathrm{c}\varphi_i & 0 \\ -\mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i + \mathrm{c}\psi_i\mathrm{c}\varphi_i & -\mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i - \mathrm{c}\psi_i\mathrm{s}\varphi_i & 0 \\ \mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i + \mathrm{s}\psi_i\mathrm{c}\varphi_i & \mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i - \mathrm{s}\psi_i\mathrm{s}\varphi_i & 0 \end{bmatrix}, \tag{29}$$

$$\left[\dot{\mathbf{A}}_i\right] = \dot\psi_i\left[\mathbf{A}_{\psi_i}\right] + \dot\theta_i\left[\mathbf{A}_{\theta_i}\right] + \dot\varphi_i\left[\mathbf{A}_{\varphi_i}\right],\ \left\{\mathbf{u}_{\psi_i}\right\} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ \left\{\mathbf{u}_{\theta_i}\right\} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \left\{\mathbf{u}_{\varphi_i}\right\} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \tag{30}$$

$$\left[\mathbf{Q}_i\right] = \left[\boldsymbol{\varphi}_i\right]^{\mathrm{T}}\left[\left[\boldsymbol{\theta}_i\right]^{\mathrm{T}}\left\{\mathbf{u}_{\psi_i}\right\}\ \left\{\mathbf{u}_{\theta_i}\right\}\ \left\{\mathbf{u}_{\varphi_i}\right\}\right] = \begin{bmatrix} \cos\varphi_i\cos\theta_i & \sin\varphi_i & 0 \\ -\sin\varphi_i\cos\theta_i & \cos\varphi_i & 0 \\ \sin\theta_i & 0 & 1 \end{bmatrix}, \qquad \left[\mathbf{Q}_{\varphi i}\right] = \begin{bmatrix} -\sin\varphi_i\cos\theta_i & \cos\varphi_i & 0 \\ -\cos\varphi_i\cos\theta_i & -\sin\varphi_i & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left[\mathbf{Q}_{\theta i}\right] = \begin{bmatrix} -\cos\varphi_i\sin\theta_i & 0 & 0 \\ \sin\varphi_i\sin\theta_i & 0 & 0 \\ \cos\theta_i & 0 & 0 \end{bmatrix}, \tag{31}$$

$$\left[\dot{\mathbf{Q}}_i\right] = \dot\varphi_i\left[\mathbf{Q}_{\varphi_i}\right] + \dot\theta\left[\mathbf{Q}_{\theta_i}\right] = \begin{bmatrix} -\dot\varphi_i\sin\varphi_i\cos\theta_i - \dot\theta_i\cos\varphi_i\sin\theta_i & \dot\varphi_i\cos\varphi_i & 0 \\ -\dot\varphi_i\cos\varphi_i\cos\theta_i + \dot\theta_i\sin\varphi_i\sin\theta_i & -\dot\varphi_i\sin\varphi_i & 0 \\ \dot\theta_i\cos\theta_i & 0 & 0 \end{bmatrix}, \tag{32}$$

$$\left\{\boldsymbol{\omega}_i\right\} = \left[Q_i\right]\begin{bmatrix} \dot\psi_i \\ \dot\theta_i \\ \dot\varphi_i \end{bmatrix} = \begin{bmatrix} \dot\psi_i\cos\varphi_i\cos\theta_i + \dot\theta_i\sin\varphi_i \\ -\dot\psi\sin\varphi_i\cos\theta_i + \dot\theta_i\cos\varphi_i \\ -\dot\psi_i\sin\theta_i - \dot\varphi_i \end{bmatrix}, \tag{33}$$

$$\left[\boldsymbol{\omega}_i\right] = \begin{bmatrix} 0 & -\dot\psi\sin\theta_i - \dot\varphi_i & -\dot\psi\sin\varphi_i\cos\theta_i + \dot\theta_i\cos\varphi_i \\ \dot\psi\sin\varphi_i + \dot\varphi_i & 0 & -\dot\psi_i\cos\varphi_i\cos\theta_i - \dot\theta_i\sin\varphi_i \\ \dot\psi\sin\varphi_i\cos\theta_i - \dot\theta_i\cos\varphi_i & \dot\psi_i\cos\varphi_i\cos\theta_i + \dot\theta_i\sin\varphi_i & 0 \end{bmatrix}, \tag{34}$$

$$\left[\mathbf{m}_i\right] = \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{bmatrix},\ \left[\mathbf{J}_{O_i}\right] = \begin{bmatrix} J_{x_i} & 0 & 0 \\ 0 & J_{y_i} & 0 \\ 0 & 0 & J_{z_i} \end{bmatrix}, \tag{35}$$

$$[\mathbf{A}_i][\mathbf{S}_i]^{\mathrm{T}}[\mathbf{Q}_i] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ [\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{S}_i][\mathbf{A}_i]^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{36}$$

$$[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{J}_{O_i}][\dot{\mathbf{Q}}_i]^{\mathrm{T}} =$$

$$= \begin{bmatrix} \begin{matrix} J_{x_i}\mathrm{c}\varphi_i\mathrm{c}\theta_i\left(\dot{\varphi}_i\mathrm{s}\varphi_i\mathrm{c}\theta_i - \dot{\theta}_i\mathrm{c}\varphi_i\mathrm{s}\theta_i\right) - \\ - J_{y_i}\mathrm{s}\varphi_i\mathrm{c}\theta_i\left(-\dot{\varphi}_i\mathrm{c}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{s}\varphi_i\mathrm{s}\theta_i\right) + J_{z_i}\mathrm{s}\theta_i\mathrm{c}\theta_i\dot{\theta}_i \end{matrix} & J_{x_i}\mathrm{c}^2\varphi_i\mathrm{c}\theta_i\dot{\varphi}_i + -J_{y_i}\mathrm{s}^2\varphi_i\mathrm{c}\theta_i\dot{\varphi}_i & 0 \\ \begin{matrix} J_{x_i}\mathrm{s}\varphi_i\left(\dot{\varphi}_i\mathrm{s}\varphi_i\mathrm{c}\theta_i - \dot{\theta}_i\mathrm{c}\varphi_i\mathrm{s}\theta_i\right) + \\ + J_{y_i}\mathrm{c}\varphi_i\left(-\dot{\varphi}_i\mathrm{c}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{s}\varphi_i\mathrm{s}\theta_i\right) \end{matrix} & J_{x_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\dot{\varphi}_i - -J_{y_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\dot{\varphi}_i & 0 \\ J_{z_i}\mathrm{c}\theta_i\dot{\theta}_i & 0 & 0 \end{bmatrix}, \tag{37}$$

$$[\mathbf{Q}_i]^{\mathrm{T}}[\boldsymbol{\omega}_i][\mathbf{J}_{O_i}][\mathbf{Q}_i] =$$

$$= \begin{bmatrix} \begin{matrix} -J_{x_i}\mathrm{c}\varphi_i\mathrm{c}\theta_i\left(\dot{\varphi}_i\mathrm{s}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{c}\varphi_i\mathrm{s}\theta_i\right) - \\ - J_{y_i}\mathrm{s}\varphi_i\mathrm{c}\theta_i\left(-\dot{\varphi}_i\mathrm{c}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{s}\varphi_i\mathrm{s}\theta_i\right) + \\ + J_{z_i}\mathrm{s}\theta_i\mathrm{c}\theta_i\dot{\theta}_i \end{matrix} & \begin{matrix} -J_{x_i}\mathrm{s}\varphi_i\left(\dot{\varphi}_i\mathrm{s}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{c}\varphi_i\mathrm{s}\theta_i\right) + \\ + J_{y_i}\mathrm{c}\varphi_i\left(-\dot{\varphi}_i\mathrm{c}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{s}\varphi_i\mathrm{s}\theta_i\right) \end{matrix} & J_{z_i}\mathrm{c}\theta_i\dot{\theta}_i \\ \begin{matrix} J_{x_i}\mathrm{c}^2\varphi_i\mathrm{c}\theta_i\left(\dot{\psi}_i\mathrm{s}\theta_i + \dot{\varphi}_i\right) - \\ - J_{y_i}\mathrm{s}^2\varphi_i\mathrm{c}\theta_i\left(-\dot{\psi}_i\mathrm{s}\theta_i - \dot{\varphi}_i\right) - \\ - J_{z_i}\mathrm{s}\theta_i\mathrm{c}\varphi_i\dot{\psi}_i \end{matrix} & \begin{matrix} J_{x_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\left(\dot{\psi}_i\mathrm{s}\theta_i + \dot{\varphi}_i\right) - \\ - J_{y_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\left(\dot{\psi}_i\mathrm{s}\theta_i + \dot{\varphi}_i\right) \end{matrix} & -J_{z_i}\mathrm{c}\theta_i\dot{\theta}_i \\ \begin{matrix} J_{x_i}\mathrm{c}\varphi_i\mathrm{c}\theta_i\left(\dot{\psi}_i\mathrm{s}\varphi_i\mathrm{c}\theta_i - \dot{\theta}_i\mathrm{c}\varphi_i\right) - \\ - J_{y_i}\mathrm{s}\varphi_i\mathrm{c}\theta_i\left(\dot{\psi}_i\mathrm{c}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{s}\varphi_i\right) \end{matrix} & \begin{matrix} J_{x_i}\mathrm{s}\varphi_i\left(\dot{\psi}_i\mathrm{s}\varphi_i\mathrm{c}\theta_i - \dot{\theta}_i\mathrm{c}\varphi_i\right) + \\ J_{y_i}\mathrm{c}\varphi_i\left(\dot{\psi}_i\mathrm{c}\varphi_i\mathrm{c}\theta_i + \dot{\theta}_i\mathrm{s}\varphi_i\right) \end{matrix} & 0 \end{bmatrix}. \tag{38}$$

We denote by $\beta_j^{(i)}$, $j = \overline{1,3}$, $i = \overline{1,2}$, the components of the vector

$$\left\{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_i}\right\} = \begin{bmatrix} \beta_1^{(i)} \\ \beta_2^{(i)} \\ \beta_3^{(i)} \end{bmatrix} = -\left[[\mathbf{Q}]^{\mathrm{T}}[\mathbf{J}_O][\dot{\mathbf{Q}}] + [\mathbf{Q}]^{\mathrm{T}}[\boldsymbol{\omega}][\mathbf{J}_O][\mathbf{Q}]\right] \begin{bmatrix} \dot{\psi}_i \\ \dot{\theta}_i \\ \dot{\varphi}_i \end{bmatrix}. \tag{39}$$

We also have

$$[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{J}_{O_i}][\mathbf{Q}_i] = \begin{bmatrix} J_{x_i}\mathrm{c}^2\varphi_i\mathrm{c}^2\theta_i + J_{y_i}\mathrm{s}^2\varphi_i\mathrm{c}^2\theta_i + J_{z_i}\mathrm{s}^2\theta_i & J_{x_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\mathrm{c}\theta_i - J_{y_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\mathrm{c}\theta_i & J_{z_i}\mathrm{s}^2\theta_i \\ J_{x_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\mathrm{c}\theta_i - J_{y_i}\mathrm{s}\varphi_i\mathrm{c}\varphi_i\mathrm{c}\theta_i & J_{x_i}\mathrm{s}^2\varphi_i + J_{y_i}\mathrm{c}^2\varphi_i & 0 \\ J_{z_i}\mathrm{s}\theta_i & 0 & J_{z_i} \end{bmatrix} \tag{40}$$

and let us denote by $\gamma_{j_k}^{(i)}$, $j,\ k = \overline{1,3}$, $i = \overline{1,2}$, the components of these matrices.

We will consider the order of parameters X_{O_1}, Y_{O_1}, Z_{O_1}, ψ_1, θ_1, φ_1, X_{O_2}, Y_{O_2}, Z_{O_2}, ψ_2, θ_2, φ_2.

We have only one constraints given by the belonging of the point C to the two bars. For the bar AC we may write $x_C = l_1/2$, $y_C = 0$, $z_C = 0$, while for the bar BC we have $x_C = -l_2/2$, $y_C = 0$, $z_C = 0$. From the relations

$$\begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} = \begin{bmatrix} X_{O_1} \\ Y_{O_1} \\ Z_{O_1} \end{bmatrix} + [\mathbf{A}] \begin{bmatrix} \frac{l_1}{2} \\ 0 \\ 0 \end{bmatrix},\ \begin{bmatrix} X_C \\ Y_C \\ Z_C \end{bmatrix} = \begin{bmatrix} X_{O_2} \\ Y_{O_2} \\ Z_{O_2} \end{bmatrix} + [\mathbf{A}] \begin{bmatrix} -\frac{l_2}{2} \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

it results the expressions

$$X_{O_1} + \frac{l_1}{2}\cos\theta_1\cos\varphi_1 = X_{O_2} - \frac{l_2}{2}\cos\theta_2\cos\varphi_2\,,$$

$$Y_{O_1} + \frac{l_1}{2}\left(\sin\psi_1\sin\theta_1\cos\varphi_1 + \cos\psi_1\sin\varphi_1\right) = Y_{O_2} - \frac{l_2}{2}\left(\sin\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2\right),$$

$$Z_{O_1} + \frac{l_1}{2}\left(-\cos\psi_1\sin\theta_1\cos\varphi_1 + \sin\psi_1\sin\varphi_1\right) = Z_{O_2} - \frac{l_2}{2}\left(-\cos\psi_2\sin\theta_2\cos\varphi_2 + \sin\psi_2\sin\varphi_2\right) \quad (42)$$

and the constraint function read

$$f_1\left(X_{O_1},\ldots,\varphi_2\right) = X_{O_1} - X_{O_2} + \frac{l_1}{2}\cos\theta_1\cos\varphi_1 + \frac{l_2}{2}\cos\theta_2\cos\varphi_2 = 0\,,$$

$$f_2\left(X_{O_1},\ldots,\varphi_2\right) = Y_{O_1} - Y_{O_2} + \frac{l_1}{2}\left(\sin\psi_1\sin\theta_1\cos\varphi_1 + \cos\psi_1\sin\varphi_1\right) +$$

$$+ \frac{l_2}{2}\left(\sin\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2\right) = 0\,,$$

$$f_3\left(X_{O_1},\ldots,\varphi_2\right) = Z_{O_1} - Z_{O_2} + \frac{l_1}{2}\left(-\cos\psi_1\sin\theta_1\cos\varphi_1 + \sin\psi_1\sin\varphi_1\right) +$$

$$+ \frac{l_2}{2}\left(-\cos\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2\right) = 0\,. \quad (43)$$

The components of the matrix of constraints are

$$B_{11} = \frac{\partial f_1}{\partial X_{O_1}} = 1\,,\quad B_{12} = \frac{\partial f_1}{\partial Y_{O_1}} = 0\,,\quad B_{13} = \frac{\partial f_1}{\partial Z_{O_1}} = 0\,,\quad B_{14} = \frac{\partial f_1}{\partial \psi_1} = 0\,,\quad B_{15} = \frac{\partial f_1}{\partial \theta_1} = -\frac{l_1}{2}\sin\theta_1\cos\varphi_1\,,$$

$$B_{16} = \frac{\partial f_1}{\partial \varphi_1} = -\frac{l_1}{2}\cos\theta_1\sin\varphi_1\,,\ B_{17} = \frac{\partial f_1}{\partial X_{O2}} = -1\,,\ B_{18} = \frac{\partial f_1}{\partial Y_{O2}} = 0\,,\ B_{19} = \frac{\partial f_1}{\partial Z_{O2}} = 0\,,\ B_{110} = \frac{\partial f_1}{\partial \psi_2} = 0\,,$$

$$B_{111} = \frac{\partial f_1}{\partial \theta_2} = -\frac{l_2}{2}\sin\theta_2\cos\varphi_2\,,\ B_{112} = \frac{\partial f_1}{\partial \varphi_2} = -\frac{l_2}{2}\cos\theta_2\sin\varphi_2 \quad (44)$$

$$B_{21} = \frac{\partial f_2}{\partial X_{O_1}} = 0\,,\quad B_{22} = \frac{\partial f_2}{\partial Y_{O_1}} = 1\,,\quad B_{23} = \frac{\partial f_2}{\partial Z_{O_1}} = 0\,,\quad B_{24} = \frac{\partial f_2}{\partial \psi_1} = \frac{l_1}{2}\left(\cos\psi_1\sin\theta_1 - \sin\psi_1\sin\varphi_1\right),$$

$$B_{25} = \frac{\partial f_2}{\partial \theta_1} = \frac{l_1}{2}\sin\psi_1\cos\theta_1\cos\varphi_1\,,\qquad B_{26} = \frac{\partial f_2}{\partial \varphi_1} = \frac{l_1}{2}\left(-\sin\psi_1\sin\theta_1\sin\varphi_1 + \cos\psi_1\cos\varphi_1\right),$$

$$B_{27} = \frac{\partial f_2}{\partial X_{O2}} = 0\,,\qquad B_{28} = \frac{\partial f_2}{\partial Y_{O2}} = -1\,,\qquad B_{29} = \frac{\partial f_2}{\partial Z_{O2}} = 0\,,$$

$$B_{210} = \frac{\partial f_2}{\partial \psi_2} = \frac{l_2}{2}\left(\cos\psi_2\sin\theta_2\cos\varphi_2 - \sin\psi_2\sin\varphi_2\right),\qquad B_{211} = \frac{\partial f_2}{\partial \theta_2} = \frac{l_2}{2}\sin\psi_2\cos\theta_2\cos\varphi_2\,,$$

$$B_{212} = \frac{\partial f_2}{\partial \varphi_2} = \frac{l_2}{2}\left(-\sin\psi_2\sin\theta_2\sin\varphi_2 + \cos\psi_2\cos\varphi_2\right), \quad (45)$$

$$B_{31} = \frac{\partial f_3}{\partial X_{O_1}} = 0\,,\qquad B_{32} = \frac{\partial f_3}{\partial Y_{O_1}} = 0\,,\qquad B_{33} = \frac{\partial f_3}{\partial Z_{O_1}} = 1\,,$$

$$B_{34} = \frac{\partial f_3}{\partial \psi_1} = \frac{l_1}{2}\left(\sin\psi_1\sin\theta_1\cos\varphi_1 + \cos\psi_1\sin\varphi_1\right),\qquad B_{35} = \frac{\partial f_2}{\partial \theta_1} = -\frac{l_1}{2}\cos\psi_1\cos\theta_1\cos\varphi_1\,,$$

$$B_{36} = \frac{\partial f_3}{\partial \varphi_1} = \frac{l_1}{2}\left(\cos\psi_1\sin\theta_1\sin\varphi_1 + \sin\psi_1\cos\varphi_1\right),\qquad B_{37} = \frac{\partial f_3}{\partial X_{O2}} = 0\,,\qquad B_{38} = \frac{\partial f_3}{\partial Y_{O2}} = 0\,,$$

$$B_{39} = \frac{\partial f_3}{\partial Z_{O2}} = -1\,,\qquad B_{310} = \frac{\partial f_3}{\partial \psi_2} = \frac{l_2}{2}\left(\sin\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2\right),$$

$$B_{311} = \frac{\partial f_3}{\partial \theta_2} = -\frac{l_2}{2}\cos\psi_2\cos\theta_2\cos\varphi_2\,,\ B_{312} = \frac{\partial f_3}{\partial \varphi_2} = \frac{l_2}{2}\left(\cos\psi_2\sin\theta_2\sin\varphi_2 + \sin\psi_2\cos\varphi_2\right). \quad (46)$$

We may also write

$$\left\{\mathbf{F}_{\mathbf{s}_1}\right\} = \left[0\ \ 0\ -m_1 g\right]^{\mathrm{T}},\quad \left\{\mathbf{F}_{\mathbf{s}_2}\right\} = \left[0\ \ 0\ -m_2 g\right]^{\mathrm{T}},\quad \left\{\mathbf{F}_{\boldsymbol{\beta}_1}\right\} = \left[0\ \ 0\ \ 0\right]^{\mathrm{T}},\quad \left\{\mathbf{F}_{\boldsymbol{\beta}_2}\right\} = \left[0\ \ 0\ \ 0\right]^{\mathrm{T}},\quad \left\{\widetilde{\mathbf{F}}_{\mathbf{s}_1}\right\} = \left[0\ \ 0\ \ 0\right]^{\mathrm{T}},$$

$$\left\{\widetilde{\mathbf{F}}_{\mathbf{s}_2}\right\} = \left[0\ \ 0\ \ 0\right]^{\mathrm{T}},\ \left\{\widetilde{\mathbf{F}}_{\boldsymbol{\beta}_1}\right\} = \left[\beta_1^{(1)}\ \ \beta_2^{(1)}\ \ \beta_3^{(1)}\right]^{\mathrm{T}},\ \left\{\widetilde{\mathbf{F}}_{\boldsymbol{\beta}_2}\right\} = \left[\beta_1^{(2)}\ \ \beta_2^{(2)}\ \ \beta_3^{(2)}\right]^{\mathrm{T}}, \quad (47)$$

$$[\mathbf{M}_i]=\begin{bmatrix} m_i & 0 & 0 & 0 & 0 & 0\\ 0 & m_i & 0 & 0 & 0 & 0\\ 0 & 0 & m_i & 0 & 0 & 0\\ 0 & 0 & 0 & \gamma_{11}^{(i)} & \gamma_{12}^{(i)} & \gamma_{13}^{(i)}\\ 0 & 0 & 0 & \gamma_{21}^{(i)} & \gamma_{22}^{(i)} & \gamma_{23}^{(i)}\\ 0 & 0 & 0 & \gamma_{31}^{(i)} & \gamma_{32}^{(i)} & \gamma_{33}^{(i)} \end{bmatrix},\qquad \{\mathbf{F}_{\mathbf{q}_1}\}=[0\ 0\ -mg_1\ 0\ 0\ 0]^{\mathrm{T}},\qquad \{\mathbf{F}_{\mathbf{q}_2}\}=[0\ 0\ -mg_2\ 0\ 0\ 0]^{\mathrm{T}},$$

$$\{\widetilde{\mathbf{F}}_{\mathbf{q}_1}\}=\left[0\ 0\ 0\ \beta_1^{(1)}\ \beta_2^{(1)}\ \beta_3^{(1)}\right]^{\mathrm{T}},\qquad \{\widetilde{\mathbf{F}}_{\mathbf{q}_2}\}=\left[0\ 0\ 0\ \beta_1^{(2)}\ \beta_2^{(2)}\ \beta_3^{(2)}\right]^{\mathrm{T}},$$

$$\{\mathbf{F}_{\mathbf{q}}\}=[0\ 0\ -m_1 g\ 0\ 0\ 0\ 0\ 0\ -m_2 g\ 0\ 0\ 0]^{\mathrm{T}},\qquad \{\widetilde{\mathbf{F}}_{\mathbf{q}}\}=\left[0\ 0\ 0\ \beta_1^{(1)}\ \beta_2^{(1)}\ \beta_3^{(1)}\ 0\ 0\ 0\ \beta_1^{(2)}\ \beta_2^{(2)}\ \beta_3^{(2)}\right]^{\mathrm{T}},$$

$$\{\mathbf{F}_{\mathbf{q}}\}+\{\widetilde{\mathbf{F}}_{\mathbf{q}}\}=\left[0\ 0\ -m_1 g\ \beta_1^{(1)}\ \beta_2^{(1)}\ \beta_3^{(1)}\ 0\ 0\ -m_2 g\ \beta_1^{(2)}\ \beta_2^{(2)}\ \beta_3^{(2)}\right]^{\mathrm{T}},\qquad \{\mathbf{C}\}=[0\ 0\ 0]^{\mathrm{T}},\qquad \{\dot{\mathbf{C}}\}=[0\ 0\ 0]^{\mathrm{T}},$$

$$\{\dot{\mathbf{B}}\}=\begin{bmatrix} \dot{B}_{11} & \dot{B}_{12} & \cdots & \dot{B}_{112}\\ \dot{B}_{21} & \dot{B}_{22} & \cdots & \dot{B}_{212}\\ \dot{B}_{31} & \dot{B}_{32} & \cdots & \dot{B}_{312} \end{bmatrix}^{\mathrm{T}},\qquad [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}=\begin{bmatrix} \dot{B}_{15}\dot{\theta}_1+\dot{B}_{15}\dot{\varphi}_1+\dot{B}_{111}\dot{\theta}_2+\dot{B}_{112}\dot{\varphi}_2\\ \dot{B}_{24}\dot{\psi}_1+\dot{B}_{25}\dot{\theta}_1+\dot{B}_{26}\dot{\varphi}_1+\dot{B}_{210}\dot{\psi}_2+\dot{B}_{211}\dot{\theta}_2+\dot{B}_{212}\dot{\varphi}_2\\ \dot{B}_{34}\dot{\psi}_1+\dot{B}_{35}\dot{\theta}_1+\dot{B}_{36}\dot{\varphi}_1+\dot{B}_{310}\dot{\psi}_2+\dot{B}_{311}\dot{\theta}_2+\dot{B}_{312}\dot{\varphi}_2 \end{bmatrix},$$

$$\{\mathbf{q}\}=\left[X_{O_1}\ Y_{O_1}\ Z_{O_1}\ \psi_1\ \theta_1\ \varphi_1\ X_{O_2}\ Y_{O_2}\ Z_{O_2}\ \psi_2\ \theta_2\ \varphi_2\right]^{\mathrm{T}},\ \{\boldsymbol{\lambda}\}=[\lambda_1\ \lambda_2\ \lambda_2]^{\mathrm{T}}. \tag{48}$$

We denote by $[\mathbf{M}]$ the fifteenth order squared matrix

$$[\mathbf{M}]=\begin{bmatrix}
m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{11} & -B_{21} & -B_{31}\\
0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{12} & -B_{22} & -B_{32}\\
0 & 0 & m_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -B_{13} & -B_{23} & -B_{33}\\
0 & 0 & 0 & \gamma_{11}^{(1)} & \gamma_{12}^{(1)} & \gamma_{13}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & -B_{14} & -B_{24} & -B_{34}\\
0 & 0 & 0 & \gamma_{21}^{(1)} & \gamma_{22}^{(1)} & \gamma_{23}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & -B_{15} & -B_{25} & -B_{35}\\
0 & 0 & 0 & \gamma_{31}^{(1)} & \gamma_{32}^{(1)} & \gamma_{33}^{(1)} & 0 & 0 & 0 & 0 & 0 & 0 & -B_{16} & -B_{26} & -B_{36}\\
0 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & 0 & -B_{17} & -B_{27} & -B_{37}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & 0 & -B_{18} & -B_{28} & -B_{38}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 & 0 & 0 & -B_{19} & -B_{29} & -B_{39}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{11}^{(2)} & \gamma_{12}^{(2)} & \gamma_{13}^{(2)} & -B_{110} & -B_{210} & -B_{310}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{21}^{(2)} & \gamma_{22}^{(2)} & \gamma_{23}^{(2)} & -B_{111} & -B_{211} & -B_{311}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{31}^{(2)} & \gamma_{31}^{(2)} & \gamma_{33}^{(2)} & -B_{112} & -B_{212} & -B_{312}\\
B_{11} & B_{12} & B_{13} & B_{14} & B_{15} & B_{16} & B_{17} & B_{18} & B_{19} & B_{110} & B_{111} & B_{112} & 0 & 0 & 0\\
B_{21} & B_{22} & B_{23} & B_{24} & B_{25} & B_{26} & B_{27} & B_{28} & B_{29} & B_{210} & B_{211} & B_{212} & 0 & 0 & 0\\
B_{31} & B_{32} & B_{33} & B_{34} & B_{35} & B_{36} & B_{37} & B_{38} & B_{39} & B_{310} & B_{311} & B_{312} & 0 & 0 & 0
\end{bmatrix} \tag{49}$$

and it results the equation of motion

$$[\overline{\mathbf{M}}]\begin{bmatrix}\{\ddot{\mathbf{q}}\}\\ \{\boldsymbol{\lambda}\}\end{bmatrix}=\begin{bmatrix}\{\mathbf{F}_{\mathbf{q}}\}+\{\widetilde{\mathbf{F}}_{\mathbf{q}}\}\\ -[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}\end{bmatrix}. \tag{50}$$

The reaction at the point C has the components

$$\lambda_1\begin{bmatrix}B_{11}\\ B_{12}\\ B_{13}\end{bmatrix}=\begin{bmatrix}\lambda_1\\ 0\\ 0\end{bmatrix},\ \lambda_2\begin{bmatrix}B_{21}\\ B_{22}\\ B_{23}\end{bmatrix}=\begin{bmatrix}0\\ \lambda_2\\ 0\end{bmatrix},\ \lambda_3\begin{bmatrix}B_{31}\\ B_{32}\\ B_{33}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \lambda_3\end{bmatrix} \tag{51}$$

or, equivalently,

$$\lambda_1\begin{bmatrix}B_{17}\\ B_{18}\\ B_{19}\end{bmatrix}=\begin{bmatrix}-\lambda_1\\ 0\\ 0\end{bmatrix},\ \lambda_2\begin{bmatrix}B_{27}\\ B_{28}\\ B_{29}\end{bmatrix}=\begin{bmatrix}0\\ -\lambda_2\\ 0\end{bmatrix},\ \lambda_3\begin{bmatrix}B_{37}\\ B_{38}\\ B_{39}\end{bmatrix}=\begin{bmatrix}0\\ 0\\ -\lambda_3\end{bmatrix}, \tag{52}$$

verifying the principle of action and reaction.
On the other hand,

$$\left[[Q_i]^{-1}\right]^{\mathrm{T}} = \frac{1}{\cos\theta_i}\begin{bmatrix} \cos\varphi_i & \sin\varphi_i\cos\theta_i & -\cos\varphi_i\sin\theta_i \\ -\sin\varphi_i & \cos\varphi_i\cos\theta_i & \sin\varphi_i\sin\theta_i \\ 0 & 0 & \cos\theta_i \end{bmatrix}, \tag{53}$$

$$\left\{\mathbf{F}^{(1)}_{G\boldsymbol{\beta}_1}\right\} = \lambda_1\begin{bmatrix} B_{14} \\ B_{15} \\ B_{16} \end{bmatrix} = -\lambda_1\frac{l_1}{2}\begin{bmatrix} 0 \\ \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 \end{bmatrix},$$

$$\left\{\mathbf{F}^{(1)}_{G\boldsymbol{\beta}_2}\right\} = \lambda_2\begin{bmatrix} B_{24} \\ B_{25} \\ B_{26} \end{bmatrix} = \lambda_2\frac{l_1}{2}\begin{bmatrix} \cos\psi_1\sin\theta_1\cos\varphi_1 - \sin\psi_1\sin\varphi_1 \\ \sin\psi_1\cos\theta_1\cos\varphi_1 \\ -\sin\psi_1\sin\theta_1\sin\varphi_1 + \cos\psi_1\cos\varphi_1 \end{bmatrix},$$

$$\left\{\mathbf{F}^{(1)}_{G\boldsymbol{\beta}_3}\right\} = \lambda_3\begin{bmatrix} B_{34} \\ B_{35} \\ B_{36} \end{bmatrix} = \lambda_3\frac{l_1}{2}\begin{bmatrix} \sin\psi_1\sin\theta_1\cos\varphi_1 + \cos\psi_1\sin\varphi_1 \\ -\cos\psi_1\cos\theta_1\cos\varphi_1 \\ \cos\psi_1\sin\theta_1\sin\varphi_1 + \sin\psi_1\cos\varphi_1 \end{bmatrix} \tag{54}$$

and it results the projections of the components of the moment of the reaction that acts upon the first body at the point C, onto the axes of the system $O_1x_1y_1z_1$

$$\begin{bmatrix} M_{x_1} \\ M_{y_1} \\ M_{z_1} \end{bmatrix} = \left[[Q_1]^{-1}\right]^{\mathrm{T}}\left\{\left\{\mathbf{F}^{(1)}_{G\boldsymbol{\beta}_1}\right\} + \left\{\mathbf{F}^{(1)}_{G\boldsymbol{\beta}_2}\right\} + \left\{\mathbf{F}^{(1)}_{G\boldsymbol{\beta}_3}\right\}\right\} = \frac{l_1}{2}\begin{bmatrix} E_{11}\cos\varphi_1 + E_{21}\sin\varphi_1\cos\theta_1 - E_{31}\cos\varphi_1\sin\theta_1 \\ -E_{11}\sin\varphi_1 + E_{21}\cos\varphi_1\cos\theta_1 + E_{31}\sin\varphi_1\sin\theta_1 \\ E_{31}\cos\theta_1 \end{bmatrix}, \tag{55}$$

where

$$\begin{aligned} E_{11} &= \lambda_2(\cos\psi_1\sin\theta_1\cos\varphi_1 - \sin\psi_1\sin\theta_1) + \lambda_3(\sin\psi_1\sin\theta_1\cos\varphi_1 + \cos\psi_1\sin\varphi_1), \\ E_{21} &= -\lambda_1\sin\theta_1\cos\varphi_1 + \lambda_2\sin\psi_1\cos\theta_1\cos\varphi_1 - \lambda_3\cos\psi_1\cos\theta_1\cos\varphi_1, \\ E_{31} &= -\lambda_1\cos\theta_1\sin\varphi_1 + \lambda_2(-\sin\psi_1\sin\theta_1\sin\varphi_1 + \cos\psi_1\cos\varphi_1) + \\ &\quad + \lambda_3(\cos\psi_1\sin\theta_1\sin\varphi_1 + \sin\psi_1\cos\varphi_1). \end{aligned} \tag{56}$$

We also have

$$\left\{\mathbf{F}^{(2)}_{G\boldsymbol{\beta}_1}\right\} = \lambda_1\begin{bmatrix} B_{110} \\ B_{111} \\ B_{112} \end{bmatrix} = -\lambda_1\frac{l_2}{2}\begin{bmatrix} 0 \\ -\sin\theta_2\cos\varphi_2 \\ \cos\theta_2\sin\varphi_2 \end{bmatrix},$$

$$\left\{\mathbf{F}^{(2)}_{G\boldsymbol{\beta}_2}\right\} = \lambda_2\begin{bmatrix} B_{210} \\ B_{211} \\ B_{212} \end{bmatrix} = \lambda_2\frac{l_2}{2}\begin{bmatrix} \cos\psi_2\sin\theta_2\cos\varphi_2 - \sin\psi_2\sin\varphi_2 \\ \sin\psi_2\cos\theta_2\cos\varphi_2 \\ -\sin\psi_2\sin\theta_2\sin\varphi_2 + \cos\psi_2\cos\varphi_2 \end{bmatrix},$$

$$\left\{\mathbf{F}^{(2)}_{G\boldsymbol{\beta}_3}\right\} = \lambda_3\begin{bmatrix} B_{310} \\ B_{311} \\ B_{312} \end{bmatrix} = \lambda_3\frac{l_2}{2}\begin{bmatrix} \sin\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2 \\ -\cos\psi_2\cos\theta_2\cos\varphi_2 \\ \cos\psi_2\sin\theta_2\sin\varphi_2 + \sin\psi_2\cos\varphi_2 \end{bmatrix} \tag{57}$$

and it results the projections of the components of the moment of the reaction that acts upon the second body at the point C, onto the axes of the system $O_2x_2y_2z_2$

$$\begin{bmatrix} M_{x_2} \\ M_{y_2} \\ M_{z_2} \end{bmatrix} = \left[[Q_2]^{-1}\right]^{T}\left\{\left\{\mathbf{F}^{(2)}_{G\boldsymbol{\beta}_1}\right\} + \left\{\mathbf{F}^{(2)}_{G\boldsymbol{\beta}_2}\right\} + \left\{\mathbf{F}^{(2)}_{G\boldsymbol{\beta}_3}\right\}\right\} = \frac{l_2}{2}\begin{bmatrix} E_{12}\cos\varphi_2 + E_{22}\sin\varphi_2\cos\theta_2 - E_{32}\cos\varphi_2\sin\theta_2 \\ -E_{12}\sin\varphi_2 + E_{22}\cos\varphi_2\cos\theta_2 + E_{32}\sin\varphi_2\sin\theta_2 \\ E_{32}\cos\theta_2 \end{bmatrix} \tag{58}$$

where

$$\begin{aligned} E_{12} &= \lambda_2(\cos\psi_2\sin\theta_2\cos\varphi_2 - \sin\psi_2\sin\theta_2) + \lambda_3(\sin\psi_2\sin\theta_2\cos\varphi_2 + \cos\psi_2\sin\varphi_2), \\ E_{22} &= -\lambda_1\sin\theta_2\cos\varphi_2 + \lambda_2\sin\psi_2\cos\theta_2\cos\varphi_2 - \lambda_3\cos\psi_2\cos\theta_2\cos\varphi_2, \\ E_{32} &= \lambda_2(-\sin\psi_2\sin\theta_2\sin\varphi_2 + \cos\psi_2\cos\varphi_2) + \\ &\quad -\lambda_1\cos\theta_2\sin\varphi_2 + \lambda_3(\cos\psi_2\sin\theta_2\sin\varphi_2 + \sin\psi_2\cos\varphi_2). \end{aligned} \tag{59}$$

3. CONCLUSION

In this paper we presented the generalization of the equation of motion for a mechanical system of an arbitrary number of rigid bodies. The reader may easily observe the procedure which extends the matrix of inertia, the matrix of constraints, and the matrix of generalized forces. The absence of the link between two bodies leads to a simpler form for the matrix equation of motion.

If the constraints are independent, then the left-hand matrix in the equation of motion is invertible (not necessary in a classical way). Discussions about this property of inversion could be found in [2], where the authors deal with the Moore-Penrose inverse for the matrix of constraints.

For the planar cases one may consider a simplified version of the method (the translation along the axis Oz, and the rotations about the axes Ox and Oy vanish). This is equivalent to consider particular initial conditions for this parameters, and no motion corresponding to them. The paper also includes a complete solved example.

REFERENCES

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