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DYNAMICS OF THE FOUR-BAR MECHANISMS WITH RRR OR RTR DYADS BY A MULTIBODY APPROACH

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Abstract: *This paper continues our previous one applying the theory developed for the case of mechanical systems with an arbitrary number of rigid bodies to the case of two very often met mechanisms. The calculation is conducted until complete solve of the problem. The equations are nonlinear and the results can be obtained only by numerical calculation.*
Keywords: *multibody, RRR dyad, RRT dyad, mechanism, equilibrium*

1. INTRODUCTION

Generalizing the equation in [1], in our previous work [2] we presented the matrix equation of motion for a mechanical system in the form

$$\begin{bmatrix} [\mathbf{M}] & -[\mathbf{B}]^{\mathrm{T}} \\ [\mathbf{B}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \{\boldsymbol{\lambda}\} \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}_{\mathbf{q}}\} + \{\tilde{\mathbf{F}}_{\mathbf{q}}\} \\ \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}, \tag{1}$$

the significance of the parameters being given in [2].
In this paper we will consider two mechanisms, very often met in the practice. The mechanisms contain a RRR, and a RTR, respectively, dyad.

2. THE FOUR-BAR MECHANISM WITH RRR DYAD

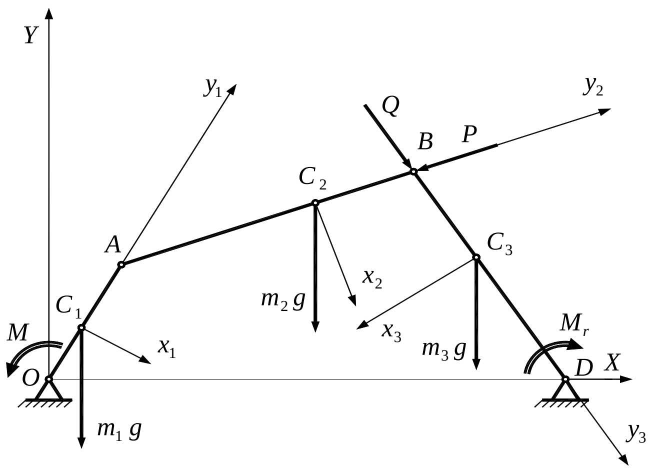

Figure 1: The four-bar mechanism with RRR dyad.

We consider the four-bar mechanism $OABD$ in Figure 1 for which one knows: the lengths of the bars $OA = l_1$, $AB = l_2$, $BD = l_3$, the positions of the centers of weight $OC_1 = r_1$, $AC_2 = r_2$, $BC_3 = r_3$, the masses m_1, m_2,

m_3, the driving moment M, the resistant moment M_r, the forces P, and Q that act along the directions of the bars AB, and BD, respectively. The mechanism work in a vertical plan OXY, the coordinates of the point D being X_D, 0, 0. The mobile system $C_i x_i y_i z_i$, $i=\overline{1,3}$, are central principal systems of inertia and one knows J_{x_i}, J_{y_i}, J_{z_i}, $i=\overline{1,3}$. One asks to be determine the motion equation of the mechanism.

We choose the Bryan rotational schema. Let us denote by X_{C_i}, Y_{C_i}, Z_{C_i}, $i=\overline{1,3}$, the coordinates of the centers of weight of the three bars. Theoretically, the system has 18 degrees of freedom, but this number decreases because the constraints that appear.
The coordinates of the point A on the bars OA and AB are

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} = \begin{bmatrix} X_{C_1} \\ Y_{C_1} \\ Z_{C_1} \end{bmatrix} + [\mathbf{A}_1] \begin{bmatrix} 0 \\ l_1 - r_1 \\ 0 \end{bmatrix} = \begin{bmatrix} X_{C_1} - (l_1 - r_1)\cos\theta_1 \sin\varphi_1 \\ Y_{C_1} + (l_1 - r_1)(-\sin\psi_1 \sin\theta_1 \sin\varphi_1 + \cos\psi_1 \cos\varphi_1) \\ Z_{C_1} + (l_1 - r_1)(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1) \end{bmatrix},$$

$$\begin{bmatrix} X_A \\ Y_A \\ Z_A \end{bmatrix} = \begin{bmatrix} X_{C_2} \\ Y_{C_2} \\ Z_{C_2} \end{bmatrix} + [\mathbf{A}_2] \begin{bmatrix} 0 \\ -r_2 \\ 0 \end{bmatrix} = \begin{bmatrix} X_{C_2} + r_2 \cos\theta_2 \sin\varphi_2 \\ Y_{C_2} - r_2(-\sin\psi_2 \sin\theta_2 \sin\varphi_2 + \cos\psi_2 \cos\varphi_2) \\ Z_{C_2} - r_2(\cos\psi_2 \sin\theta_2 \sin\varphi_2 + \sin\psi_2 \cos\varphi_2) \end{bmatrix}, \tag{2}$$

the first functions of constraint being

$$\begin{aligned} f_1(\mathbf{q}) &= X_{C_1} - (l_1 - r_1)\cos\theta_1 \sin\varphi_1 - X_{C_2} - r_2 \cos\theta_2 \sin\varphi_2 = 0, \\ f_2(\mathbf{q}) &= Y_{C_1} + (l_1 - r_1)(-\sin\psi_1 \sin\theta_1 \sin\varphi_1 + \cos\psi_1 \cos\varphi_1) - \\ &\quad - Y_{C_2} + r_2(-\sin\psi_2 \sin\theta_2 \sin\varphi_2 + \cos\psi_2 \cos\varphi_2) = 0, \\ f_3(\mathbf{q}) &= Z_{C_1} + (l_1 - r_1)(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1) - \\ &\quad - Z_{C_2} + r_2(\cos\psi_2 \sin\theta_2 \sin\varphi_2 + \sin\psi_2 \cos\varphi_2) = 0, \end{aligned} \tag{3}$$

where $\{\mathbf{q}\} = \begin{bmatrix} X_{C_1} & Y_{C_1} & \ldots & \theta_1 & \varphi_1 & X_{C_2} & \ldots & \varphi_2 & X_{C_3} & \ldots & \varphi_3 \end{bmatrix}^{\mathrm{T}}$.
Since the point A is in the OXY, it also results the constraint function

$$f_4(\mathbf{q}) = Z_{C_1} + (l_1 - r_1)(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1) = 0. \tag{4}$$

Similarly, for the point B we get

$$\begin{aligned} f_5(\mathbf{q}) &= X_{C_2} - (l_2 - r_2)\cos\theta_2 \sin\varphi_2 - X_{C_3} - r_3 \cos\theta_3 \sin\varphi_3 = 0, \\ f_6(\mathbf{q}) &= Y_{C_2} + (l_2 - r_2)(-\sin\psi_2 \sin\theta_2 \sin\varphi_2 + \cos\psi_2 \cos\varphi_2) - \\ &\quad - Y_{C_3} + r_3(-\sin\psi_3 \sin\theta_3 \sin\varphi_3 + \cos\psi_3 \cos\varphi_3) = 0, \\ f_7(\mathbf{q}) &= Z_{C_2} + (l_2 - r_2)(\cos\psi_2 \sin\theta_2 \sin\varphi_2 + \sin\psi_2 \cos\varphi_2) - \\ &\quad - Z_{C_3} + r_3(\cos\psi_3 \sin\theta_3 \sin\varphi_3 + \sin\psi_3 \cos\varphi_3) = 0, \\ f_8(\mathbf{q}) &= Z_{C_2} + (l_2 - r_2)(\cos\psi_2 \sin\theta_2 \sin\varphi_2 + \sin\psi_2 \cos\varphi_2) = 0. \end{aligned} \tag{5}$$

For the point O we have

$$\begin{aligned} f_9(\mathbf{q}) &= X_{C_1} - r_1 \cos\theta_1 \sin\varphi_1 = 0, \qquad f_{10}(\mathbf{q}) = Y_{C_1} - r_1(-\sin\psi_1 \sin\theta_1 \sin\varphi_1 + \cos\psi_1 \cos\varphi_1) = 0, \\ f_{11}(\mathbf{q}) &= Z_{C_1} - r_1(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1) = 0. \end{aligned} \tag{6}$$

while for the point D one obtains the constraint function

$$\begin{aligned} f_{12}(\mathbf{q}) &= X_{C_3} - (l_3 - r_3)\cos\theta_3 \sin\varphi_3 - X_D = 0 \\ f_{13}(\mathbf{q}) &= Y_{C_3} + (l_3 - r_3)(\sin\psi_3 \sin\theta_3 \sin\varphi_3 + \cos\psi_3 \cos\varphi_3) = 0, \\ f_{14}(\mathbf{q}) &= Z_{C_3} + (l_3 - r_3)(\cos\psi_3 \sin\theta_3 \sin\varphi_3 + \sin\psi_3 \cos\varphi_3) = 0. \end{aligned} \tag{7}$$

We denote $b_{i,1} = \dfrac{\partial f_i}{\partial X_{C_1}}$, $b_{i,2} = \dfrac{\partial f_i}{\partial Y_{C_1}}$, …, $b_{i,6} = \dfrac{\partial f_i}{\partial \varphi_1}$, …, $b_{i,18} = \dfrac{\partial f_i}{\partial \varphi_3}$, $i=\overline{1,14}$, the matrix of constraints being

$$[\mathbf{B}] = \begin{bmatrix} b_{1,1} & b_{1,2} & \ldots & b_{1,18} \\ b_{2,1} & b_{2,2} & \ldots & b_{2,18} \\ \ldots & \ldots & \ldots & \ldots \\ b_{14,1} & b_{14,2} & \ldots & b_{14,18} \end{bmatrix}. \tag{8}$$

One obtains

$$[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} = \begin{bmatrix} \dot{b}_{1,5}\dot{\theta}_1 + \dot{b}_{1,6}\dot{\varphi}_1 + \dot{b}_{1,11}\dot{\theta}_2 + \dot{b}_{1,12}\dot{\varphi}_2 \\ \dot{b}_{2,4}\dot{\psi}_1 + \dot{b}_{2,5}\dot{\theta}_1 + \dot{b}_{2,6}\dot{\varphi}_1 + \dot{b}_{2,10}\dot{\psi}_2 + \dot{b}_{2,11}\dot{\theta}_2 + \dot{b}_{2,12}\dot{\varphi}_2 \\ \dot{b}_{3,4}\dot{\psi}_1 + \dot{b}_{3,5}\dot{\theta}_1 + \dot{b}_{3,6}\dot{\varphi}_1 + \dot{b}_{3,10}\dot{\psi}_2 + \dot{b}_{3,11}\dot{\theta}_2 + \dot{b}_{3,12}\dot{\varphi}_2 \\ \dot{b}_{4,4}\dot{\psi}_1 + \dot{b}_{4,5}\dot{\theta}_1 + \dot{b}_{4,6}\dot{\varphi}_1 \\ \dot{b}_{5,11}\dot{\theta}_2 + \dot{b}_{5,12}\dot{\varphi}_2 + \dot{b}_{5,17}\dot{\theta}_3 + \dot{b}_{2,10}\dot{\psi}_2 + \dot{b}_{5,18}\dot{\varphi}_3 \\ \dot{b}_{6,10}\dot{\psi}_2 + \dot{b}_{6,11}\dot{\theta}_2 + \dot{b}_{6,12}\dot{\varphi}_2 + \dot{b}_{6,16}\dot{\psi}_3 + \dot{b}_{6,17}\dot{\theta}_3 + \dot{b}_{6,18}\dot{\varphi}_3 \\ \dot{b}_{7,10}\dot{\psi}_2 + \dot{b}_{7,11}\dot{\theta}_2 + \dot{b}_{7,12}\dot{\varphi}_2 + \dot{b}_{7,16}\dot{\psi}_3 + \dot{b}_{7,17}\dot{\theta}_3 + \dot{b}_{7,18}\dot{\varphi}_3 \\ \dot{b}_{8,10}\dot{\psi}_2 + \dot{b}_{8,11}\dot{\theta}_2 + \dot{b}_{8,12}\dot{\varphi}_2 \\ \dot{b}_{9,5}\dot{\theta}_1 + \dot{b}_{9,6}\dot{\varphi}_1 \\ \dot{b}_{10,4}\dot{\psi}_1 + \dot{b}_{10,5}\dot{\theta}_1 + \dot{b}_{10,6}\dot{\varphi}_1 \\ \dot{b}_{11,4}\dot{\psi}_1 + \dot{b}_{11,5}\dot{\theta}_1 + \dot{b}_{11,6}\dot{\varphi}_1 \\ \dot{b}_{12,17}\dot{\theta}_3 + \dot{b}_{12,18}\dot{\varphi}_3 \\ \dot{b}_{13,16}\dot{\psi}_3 + \dot{b}_{13,17}\dot{\theta}_3 + \dot{b}_{13,18}\dot{\varphi}_3 \\ \dot{b}_{14,16}\dot{\psi}_3 + \dot{b}_{14,17}\dot{\theta}_3 + \dot{b}_{14,18}\dot{\varphi}_3 \end{bmatrix}. \quad (9)$$

We also find

$$\{\mathbf{F}_{\mathbf{s}_1}\} = [0 \ -m_1 g \ 0]^{\mathrm{T}}, \ \{\mathbf{M}_1\} = [0 \ 0 \ M]^{\mathrm{T}}, \ \{\mathbf{F}_{\boldsymbol{\beta}_1}\} = [\mathbf{Q}_1]^{\mathrm{T}}\{\mathbf{M}_1\} = \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix}, \quad (10)$$

$$\{\mathbf{F}_{\mathbf{s}_2}\} = \begin{bmatrix} \frac{P(X_A - X_B)}{l_2} & \frac{P(Y_A - Y_B)}{l_2} - m_2 g & 0 \end{bmatrix}^{\mathrm{T}}, \ \{\mathbf{F}_{\boldsymbol{\beta}_2}\} = [0 \ 0 \ 0]^{\mathrm{T}} \quad (11)$$

$$\{\mathbf{F}_{\mathbf{s}_3}\} = \begin{bmatrix} \frac{Q(X_D - X_B)}{l_3} & \frac{P(Y_D - Y_B)}{l_3} - m_3 g & 0 \end{bmatrix}^{\mathrm{T}}, \ \{\mathbf{M}_3\} = [0 \ 0 \ -M_r]^{\mathrm{T}}, \ \{\mathbf{F}_{\boldsymbol{\beta}_3}\} = [\mathbf{Q}_3]^{\mathrm{T}}\{\mathbf{M}_3\} = \begin{bmatrix} 0 \\ 0 \\ -M_r \end{bmatrix} \quad (12)$$

$$\{\tilde{\mathbf{F}}_{\mathbf{s}_i}\} = -\left[[\mathbf{A}_i][\mathbf{S}_i]^{\mathrm{T}}[\dot{\mathbf{Q}}_i] + [\dot{\mathbf{A}}_i][\mathbf{S}_i]^{\mathrm{T}}[\mathbf{Q}_i]\right]^{\mathrm{T}} \begin{bmatrix} \dot{\psi}_i \\ \dot{\theta}_i \\ \dot{\varphi}_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (13)$$

$$\{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_i}\} = -\left[[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{J}_{C_i}][\dot{\mathbf{Q}}_i] + [\mathbf{Q}_i]^{\mathrm{T}}[\boldsymbol{\omega}_i][\mathbf{J}_{C_i}][\mathbf{Q}_i]\right] \begin{bmatrix} \dot{\psi}_i \\ \dot{\theta}_i \\ \dot{\varphi}_i \end{bmatrix}, \quad (14)$$

$$[\mathbf{M}] = \begin{bmatrix} [\mathbf{M}_1] & [\mathbf{0}_{6,6}] & [\mathbf{0}_{6,6}] \\ [\mathbf{0}_{6,6}] & [\mathbf{M}_2] & [\mathbf{0}_{6,6}] \\ [\mathbf{0}_{6,6}] & [\mathbf{0}_{6,6}] & [\mathbf{M}_3] \end{bmatrix}, \quad (15)$$

$$\{\mathbf{F}_{\mathbf{q}}\} = \left[\{\mathbf{F}_{\mathbf{s}_1}\}^{\mathrm{T}} \ \{\mathbf{F}_{\boldsymbol{\beta}_1}\}^{\mathrm{T}} \ \{\mathbf{F}_{\mathbf{s}_2}\}^{\mathrm{T}} \ \{\mathbf{F}_{\boldsymbol{\beta}_2}\}^{\mathrm{T}} \ \{\mathbf{F}_{\mathbf{s}_3}\}^{\mathrm{T}} \ \{\mathbf{F}_{\boldsymbol{\beta}_3}\}^{\mathrm{T}}\right]^{\mathrm{T}}, \quad (16)$$

$$\{\tilde{\mathbf{F}}_{\mathbf{q}}\} = \left[\{\tilde{\mathbf{F}}_{\mathbf{s}_1}\}^{\mathrm{T}} \ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_1}\}^{\mathrm{T}} \ \{\mathbf{F}_{\mathbf{s}_2}\}^{\mathrm{T}} \ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_2}\}^{\mathrm{T}} \ \{\mathbf{F}_{\mathbf{s}_3}\}^{\mathrm{T}} \ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_3}\}^{\mathrm{T}}\right]^{\mathrm{T}} \quad (17)$$

and the equation of motion

$$\begin{bmatrix} [\mathbf{M}] & -[\mathbf{B}]^{\mathrm{T}} \\ [\mathbf{B}] & [\mathbf{0}_{14,14}] \end{bmatrix} \begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{14} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}_{\mathbf{q}}\} & \{\tilde{\mathbf{F}}_{\mathbf{q}}\} \\ -[\dot{\mathbf{B}}] & \{\mathbf{q}\} \end{bmatrix}. \quad (18)$$

3. THE FOUR-BAR MECHANISM WITH RTR DYAD

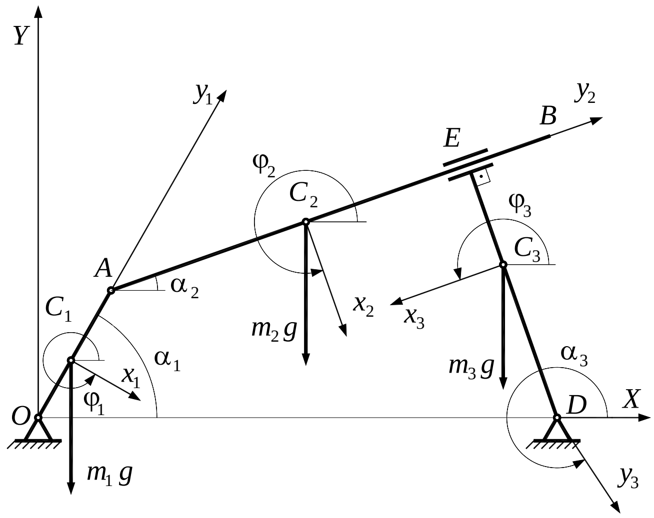

Figure 2: The four-bar mechanism with RTR dyad.

One considers the mechanism in Figure 2 for which one knows the lengths of the bars $OA = l_1$, $AB = l_2$, $ED = l_3$, the centers of weight of the three bars C_1, C_2, C_3, the positions of these centers of weight given by $OC_1 = r_1$, $AC_2 = r_2$, $EC_3 = r_3$. The point O has the coordinates 0, 0, 0, while the point D has the coordinates X_D, 0, 0. One asks to be determine the equilibrium position of the mechanism knowing that it works in vertical plan and is acted only by the weights of the bars. In addition, the local reference systems $C_i x_i y_i z_i$ are central principal systems of inertia and J_{x_i}, J_{y_i}, J_{z_i}, $i = \overline{1,3}$, are known.

We choose the Bryan rotational schema. Let X_{C_i}, Y_{C_i}, Z_{C_i}, $i = \overline{1,3}$, be the coordinates of the centers of weight of the three bars relative to the fixed reference system $OXYZ$. Again, theoretically, the system has 18 degrees of freedom.

The point A offers the first three functions of constraints

$$\begin{aligned} f_1(\mathbf{q}) &= X_{C_1} - (l_1 - r_1)\cos\theta_1 \sin\varphi_1 - X_{C_2} - r_2 \cos\theta_2 \sin\varphi_2 = 0, \\ f_2(\mathbf{q}) &= Y_{C_1} + (l_1 - r_1)(-\sin\psi_1 \sin\theta_1 \sin\varphi_1 + \cos\psi_1 \cos\varphi_1) - \\ &\quad - Y_{C_2} + r_2(-\sin\psi_2 \sin\theta_2 \sin\varphi_2 + \cos\psi_2 \cos\varphi_2) = 0, \\ f_3(\mathbf{q}) &= Z_{C_1} + (l_1 - r_1)(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1) - \\ &\quad - Z_{C_2} + r_2(\cos\psi_2 \sin\theta_2 \sin\varphi_2 + \sin\psi_2 \cos\varphi_2) = 0, \end{aligned} \tag{19}$$

where $\{\mathbf{q}\} = \begin{bmatrix} X_{C_1} & Y_{C_1} & \ldots & \theta_1 & \varphi_1 & X_{C_2} & \ldots & \varphi_2 & X_{C_3} & \ldots & \varphi_3 \end{bmatrix}^{\mathrm{T}}$.

Since the point A is in the plan OXY, we find

$$f_4(\mathbf{q}) = Z_{C_1} + (l_1 - r_1)(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1) = 0. \tag{20}$$

Knowing that the point O has the coordinates 0, 0, 0, we get

$$\begin{aligned} f_5(\mathbf{q}) &= X_{C_1} + r_1 \cos\theta_1 \sin\varphi_1 = 0, \qquad f_6(\mathbf{q}) = Y_{C_1} - r_1(-\sin\psi_1 \sin\theta_1 \sin\varphi_1 + \cos\psi_1 \cos\varphi_1), \\ f_7(\mathbf{q}) &= Z_{C_1} - r_1(\cos\psi_1 \sin\theta_1 \sin\varphi_1 + \sin\psi_1 \cos\varphi_1). \end{aligned} \tag{21}$$

Similarly, we find for the point D

$$\begin{aligned} f_8(\mathbf{q}) &= X_{C_3} - (l_3 - r_3)\cos\theta_3 \sin\varphi_3 - X_D = 0, \\ f_9(\mathbf{q}) &= Y_{C_3} + (l_3 - r_3)(-\sin\psi_3 \sin\theta_3 \sin\varphi_3 + \cos\psi_3 \cos\varphi_3) = 0, \\ f_{10}(\mathbf{q}) &= Z_{C_3} + (l_3 - r_3)(\cos\psi_3 \sin\theta_3 \sin\varphi_3 + \sin\psi_3 \cos\varphi_3) = 0, \end{aligned} \tag{22}$$

for the point B

$$f_{11}(\mathbf{q}) = Z_{C_2} + (l_2 - r_2)(\cos\psi_2 \sin\theta_2 \sin\varphi_2 + \sin\psi_2 \cos\varphi_2) = 0. \tag{23}$$

and for the point E

$$f_{12}(\mathbf{q}) = Z_{C_3} - r_3(\cos\psi_3 \sin\theta_3 \sin\varphi_3 + \sin\psi_3 \cos\varphi_3) = 0, \tag{24}$$

$$f_{13}(\mathbf{q}) = -\lfloor X_{C_3} - X_{C_2} + r_3 \cos\theta_3 \sin\varphi_3 + (l_2 - r_2)\cos\theta_2 \sin\varphi_2 \rfloor \cdot$$
$$\cdot\, l_2(-\sin\psi_2 \sin\theta_2 \sin\varphi_2 + \cos\psi_2 \cos\varphi_2) - \tag{25}$$
$$-\left[Y_{C_3} - Y_{C_2} - r_3(-\sin\psi_3 \sin\theta_3 \sin\varphi_3 + \cos\psi_3 \cos\varphi_3)\right] l_2 \cos\theta_2 \sin\varphi_2 = 0 .$$

The last function of constraint results from the condition that the vectors **AB** and **ED** are perpendicular, that is $\mathbf{AB}\cdot\mathbf{ED} = 0$. One gets

$$f_{14}(\mathbf{q}) = l_2 l_3 \cos\theta_2 \sin\varphi_2 \cos\theta_3 \sin\varphi_3 +$$
$$+ l_2 l_3(-\sin\psi_2 \sin\theta_2 \sin\varphi_2)(-\sin\psi_3 \sin\theta_3 \sin\varphi_3 + \cos\psi_3 \cos\varphi_3) = 0 . \tag{26}$$

The matrix of constraints $[\mathbf{B}]$ is formed with the elements $b_{i,j} = \dfrac{\partial f_i}{\partial q_j}$, $i = \overline{1,14}$, $j = \overline{1,18}$.

We also find

$$\{\mathbf{F}_{\mathbf{s}_1}\} = [0\ \ -m_1 g\ \ 0]^{\mathrm{T}},\ \{\mathbf{F}_{\mathbf{s}_2}\} = [0\ \ -m_2 g\ \ 0]^{\mathrm{T}},\ \{\mathbf{F}_{\mathbf{s}_3}\} = [0\ \ -m_3 g\ \ 0]^{\mathrm{T}}, \tag{27}$$

$$\{\tilde{\mathbf{F}}_{\mathbf{s}_i}\} = -\left[[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{J}_{Ci}][\dot{\mathbf{Q}}_i] + [\mathbf{Q}_i]^{\mathrm{T}}[\boldsymbol{\omega}_i][\mathbf{J}_{Ci}][\mathbf{Q}_i]\right]\begin{bmatrix}\dot{\psi}_i\\ \dot{\theta}_i\\ \dot{\varphi}_i\end{bmatrix}, \tag{28}$$

$$\{\mathbf{F}_{\boldsymbol{\beta}_i}\} = [0\ \ 0\ \ 0]^{\mathrm{T}},\ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_i}\} = -\left[[\mathbf{A}_i][\mathbf{S}_i]^{\mathrm{T}}[\dot{\mathbf{Q}}_i] + [\dot{\mathbf{A}}_i][\mathbf{S}_i]^{\mathrm{T}}[\mathbf{Q}_i]\right]^{\mathrm{T}}\begin{bmatrix}\dot{\psi}_i\\ \dot{\theta}_i\\ \dot{\varphi}_i\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \tag{29}$$

$$\{\mathbf{F}_{\mathbf{q}}\} = \left[\{\mathbf{F}_{\mathbf{s}_1}\}^{\mathrm{T}}\ \{\mathbf{F}_{\boldsymbol{\beta}_1}\}^{\mathrm{T}}\ \{\mathbf{F}_{\mathbf{s}_2}\}^{\mathrm{T}}\ \{\mathbf{F}_{\boldsymbol{\beta}_2}\}^{\mathrm{T}}\ \{\mathbf{F}_{\mathbf{s}_3}\}^{\mathrm{T}}\ \{\mathbf{F}_{\boldsymbol{\beta}_3}\}^{\mathrm{T}}\right]^{\mathrm{T}}, \tag{30}$$

$$\{\tilde{\mathbf{F}}_{\mathbf{q}}\} = \left[\{\tilde{\mathbf{F}}_{\mathbf{s}_1}\}^{\mathrm{T}}\ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_1}\}^{\mathrm{T}}\ \{\mathbf{F}_{\mathbf{s}_2}\}^{\mathrm{T}}\ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_2}\}^{\mathrm{T}}\ \{\mathbf{F}_{\mathbf{s}_3}\}^{\mathrm{T}}\ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_3}\}^{\mathrm{T}}\right]^{\mathrm{T}},\ \{\mathbf{C}\} = \{\mathbf{0}_{14,1}\}. \tag{31}$$

The moving equation reads

$$\begin{bmatrix}[\mathbf{M}] & -[\mathbf{B}]^{\mathrm{T}}\\ [\mathbf{B}] & [\mathbf{0}_{14,14}]\end{bmatrix}\begin{bmatrix}\{\ddot{\mathbf{q}}\}\\ \begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_{14}\end{bmatrix}\end{bmatrix} = \begin{bmatrix}\{\mathbf{F}_{\mathbf{q}}\} + \{\tilde{\mathbf{F}}_{\mathbf{q}}\}\\ \{\dot{\mathbf{C}}\} - [\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\}\end{bmatrix}, \tag{32}$$

and at the equilibrium we have $\{\dot{\mathbf{q}}\} = \{\mathbf{0}_{18,1}\}$, $\{\ddot{\mathbf{q}}\} = \{\mathbf{0}_{18,1}\}$, wherefrom

$$-[\mathbf{B}]^{\mathrm{T}}\begin{bmatrix}\lambda_1\\ \vdots\\ \lambda_{14}\end{bmatrix} = \{\mathbf{F}_{\mathbf{q}}\}. \tag{33}$$

On the other hand, since the system has 14 constraints, it would result that it has 4 degrees of freedom, 3 of them being useless (the rotations about the axes of coordinates $C_i y_i$, $i = \overline{1,3}$.

The equations (33) take the form

$$-\lambda_1 - \lambda_5 = 0,\ -\lambda_2 - \lambda_6 = -m_1 g,\ -\lambda_3 - \lambda_4 - \lambda_7 = 0,\ -b_{2,4}\lambda_2 - b_{3,4}\lambda_3 - b_{4,4}\lambda_4 - b_{6,4}\lambda_6 - b_{7,4}\lambda_7 = 0,$$
$$-b_{1,5}\lambda_1 - b_{2,5}\lambda_2 - b_{3,5}\lambda_3 - b_{4,5}\lambda_4 - b_{5,5}\lambda_5 - b_{6,5}\lambda_6 - b_{7,5}\lambda_7 = 0,$$
$$-b_{1,6}\lambda_1 - b_{2,6}\lambda_2 - b_{3,6}\lambda_3 - b_{4,6}\lambda_4 - b_{5,6}\lambda_5 - b_{6,6}\lambda_6 - b_{7,6}\lambda_7 = 0,\qquad -\lambda_1 - b_{13,7}\lambda_{13} = 0,$$
$$-\lambda_2 - b_{13,8}\lambda_{13} = -m_2 g,\qquad -\lambda_3 - \lambda_{11} = 0\qquad -b_{2,10}\lambda_2 - b_{3,10}\lambda_3 - b_{11,10}\lambda_{11} - b_{13,10}\lambda_{13} - b_{14,10}\lambda_{14} = 0,$$
$$-b_{1,11}\lambda_1 - b_{2,11}\lambda_2 - b_{3,11}\lambda_3 - b_{11,11}\lambda_{11} - b_{13,11}\lambda_{13} - b_{14,11}\lambda_{14} = 0,$$
$$-b_{1,12}\lambda_1 - b_{2,12}\lambda_2 - b_{3,12}\lambda_3 - b_{11,12}\lambda_{11} - b_{13,12}\lambda_{13} - b_{14,12}\lambda_{14} = 0,\qquad -\lambda_8 - b_{13,13}\lambda_{13} = 0,$$
$$-\lambda_9 - b_{13,14}\lambda_{13} = -m_3 g,\qquad -\lambda_{10} - \lambda_{12} = 0,\qquad -b_{9,16}\lambda_9 - b_{10,16}\lambda_{10} - b_{12,16}\lambda_{12} - b_{13,16}\lambda_{13} - b_{14,16}\lambda_{14} = 0,$$
$$-b_{8,17}\lambda_8 - b_{9,17}\lambda_9 - b_{10,17}\lambda_{10} - b_{12,17}\lambda_{12} - b_{13,17}\lambda_{13} - b_{14,17}\lambda_{14} = 0,$$
$$-b_{8,18}\lambda_8 - b_{9,18}\lambda_9 - b_{10,18}\lambda_{10} - b_{12,18}\lambda_{12} - b_{13,18}\lambda_{13} - b_{14,18}\lambda_{14} = 0. \tag{34}$$

Due to the three useless degrees of freedom, we can choose the initial conditions so that $\theta_1 = \theta_2 = \theta_3 = 0$. Since the motion is in the OXY we may also consider $\psi_1 = \psi_2 = \psi_3 = 0$.

It results the system

$$-\lambda_1 - \lambda_5 = 0\,, \quad -\lambda_2 - \lambda_6 = -m_1 g\,, \quad -\lambda_3 - \lambda_4 - \lambda_7 = 0\,, \quad -b_{3,4}\lambda_3 - b_{4,4}\lambda_4 - b_{7,4}\lambda_7 = 0\,,$$
$$-b_{3,5}\lambda_3 - b_{4,5}\lambda_4 - b_{7,5}\lambda_7 = 0\,, \quad -b_{1,6}\lambda_1 - b_{6,6}\lambda_6 - b_{5,6}\lambda_5 - b_{6,6}\lambda_6 = 0\,, \quad \lambda_1 - b_{13,7}\lambda_{13} = 0\,,$$
$$\lambda_2 - b_{13,8}\lambda_{13} = -m_2 g\,, \quad \lambda_3 - \lambda_{11} = 0\,, \quad -b_{3,10}\lambda_3 - b_{11,10}\lambda_{11} = 0\,, \quad -b_{3,11}\lambda_3 - b_{11,11}\lambda_{11} = 0\,,$$
$$-b_{1,12}\lambda_1 - b_{2,12}\lambda_2 - b_{3,12}\lambda_3 - b_{13,12}\lambda_{13} - b_{14,12}\lambda_{14} = 0\,, \quad \lambda_8 - b_{13,13}\lambda_{13} = 0\,, \quad \lambda_9 - b_{13,14}\lambda_{13} = -m_3 g\,,$$
$$\lambda_{10} - \lambda_{12} = 0\,, \quad -b_{10,16}\lambda_{10} - b_{12,16}\lambda_{12} = 0\,, \quad -b_{10,17}\lambda_{10} - b_{12,17}\lambda_{12} = 0\,,$$
$$-b_{8,18}\lambda_8 - b_{9,18}\lambda_9 - b_{13,18}\lambda_{13} - b_{14,18}\lambda_{14} = 0\,. \tag{35}$$

In the general case, we get $\lambda_3 = 0$, $\lambda_4 = 0$, $\lambda_7 = 0$, $\lambda_{10} = 0$, $\lambda_{11} = 0$, $\lambda_{12} = 0$, and the system

$$\lambda_1 + \lambda_5 = 0\,, \quad \lambda_2 + \lambda_6 = m_1 g\,, \quad b_{1,6}\lambda_1 + b_{6,6}\lambda_6 + b_{5,6}\lambda_5 + b_{6,6}\lambda_6 = 0\,, \quad \lambda_1 - b_{13,7}\lambda_{13} = 0\,,$$
$$\lambda_2 - b_{13,8}\lambda_{13} = -m_2 g\,, \quad b_{1,12}\lambda_1 + b_{2,12}\lambda_2 + b_{3,12}\lambda_3 + b_{13,12}\lambda_{13} + b_{14,12}\lambda_{14} = 0\,, \quad \lambda_8 - b_{13,13}\lambda_{13} = 0\,,$$
$$\lambda_9 - b_{13,14}\lambda_{13} = -m_3 g\,, \; b_{8,18}\lambda_8 + b_{9,18}\lambda_9 + b_{13,18}\lambda_{13} + b_{14,18}\lambda_{14} = 0\,, \tag{36}$$

that is a system of 9 equations with 9 unknowns (λ_1, λ_2, λ_5, λ_6, λ_8, λ_9, λ_{13}, λ_{14} and φ_1).

Between the variables φ_2, φ_3, and φ_1 there exist the relations

$$AE\cos\varphi_3 = -l_1\sin\varphi_1 + l_3\sin\varphi_3 - X_D\,, \quad AE\sin\varphi_3 = l_1\cos\varphi_1 + l_3\cos\varphi_3\,,$$
$$-l_1\sin\varphi_1\sin\varphi_3 - l_3\sin^2\varphi_3 - X_D\sin\varphi_3 = l_1\cos\varphi_1\cos\varphi_3 + l_3\cos^2\varphi_3\,,$$
$$l_1\cos\varphi_1\cos\varphi_3 + \left(l_1\sin\varphi_1 + X_D\right)\sin\varphi_3 + l_3 = 0\,. \tag{37}$$

4. CONCLUSION

In this paper we presented two applications of the theory developed in our previous work. The examples consist in two four-bar mechanism with RRR or RTR dyad. For the first case we determined the matrix equation of motion, while for the second we obtained the system of equations from which one deduces the possible positions of equilibrium.

One may observe that the great task of the method is to obtain the matrix of constraints for each case. The existence of the useless degrees of freedom leads to possible singular matrix of inertia (see the discussion at the end of [2]). In our situation, this inconvenient may be avoided by choosing particular initial conditions and assuming that there is no motion corresponding to the useless degrees of freedom.

As it was presented, the method is more difficult than the classic one for the planar mechanisms. One may reduce the complexity of the method considering directly the planar case. In this situation, the mass matrix reduces to a third order square one.

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