6th International Conference
ðComputational Mechanics and Virtual Engineering ð
COMET 2015
15-16 October 2015, Braşov, Romania

THE ANALYSIS OF CYCLICALLY SYMMETRIC STRUCTURES – HISTORICAL REMARKS

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Abstract: *This paper aims to review and explain the path of evolution of the concepts and techniques to analyze symmetric structures. First domes designed by Schwedler imply a new theory of spatial truss system in the static analysis of determinate and indeterminate problems. The system of equilibrium equations obtained has a circulant determinant. This specific form conducted to complex numbers and some savings appeared. The development of methods to exploit the symmetry properties have depended on achievements in mathematics and informatics. The use of electronic computers within the development of finite element structural analysis package had a spectacular increased at work done in this area. Two main advanced approaches in the analysis of symmetric structures have used: the discrete Fourier transformation (DFT) for rotational symmetry, and group representation theory to exploit any type of symmetry.*
Keywords: *cyclic symmetry, history, Schwedler domes, spectral decomposition, group representation theory*

1. INTRODUCTION

The author of these notes will present his point of view and his conviction about what we can be seen and learned from the history of cyclic symmetry implication in structural analysis. Obviously, it is not possible to mention explicitly all of the important results and their authors. The number of approaches and contributions in this field after 1990 is overwhelming. As a consequence of industrial progress and technologies, the addresses time is divided into periods. One covers the beginnings of the use of steel in constructions like the domes, till the era of digital computers, and one after it.

2. SYMMETRY AND CYCLIC SYMMETRY

Symmetry commonly conveys the idea of harmony and proportion. Symmetry is a special kind of transformation - a way to move an object. In plane geometry, symmetry denotes a balance of the parts of a figure to a central point, line, or plane. Axial *symmetry* is a case in which a figure can be divided by a line into two mirror-image halves. Another common type is *radial symmetry*, in which a figure can be made to coincide with itself if it is rotated about a point. Thus, a symmetric structure is a structure that is left unaltered, geometrically and mechanically, after a symmetry operation. These operations may be reflections, rotations, improper rotation (an improper rotation is a reflection in a plane, followed by a rotation about an axis perpendicular to the plane), translations or dilations/contractions. [1].

3. BEGINNINGS IN THE LATE OF THE 19TH CENTURY

3.1. Golden age of Schwedler dome

Johann Wilhelm Schwedler (1823-1894) was one of the most important Prussian civil engineers of the 19th century. Starting from 1851, Schwedler developed a new chapter of structural analysis: the truss theory. Karl Culmann (1821-1881), Squire Whipple (1804-1888) and Dimitry Ivanovich Jouravski (1821-1891) contributed to this theory, but the truss theory was confined to the late 19th century at the plane systems. Spatial structural systems of buildings such as industrial buildings, railway stations and bridges had an orthogonal structure, so that a breakdown in planar systems was sufficient. Added to this was that the spatial engineering thinking was trained since the beginning of the 19th century by the Descriptive Geometry, where the dominant method was the orthogonal projection put in the form of technical drawings. Schwedler was the first to exceed this stage with the extraordinary strength and clarity of his spatial intuition - named "*stereometric imagination*". Here only two Schwedler domes may be mentioned, which can be admired today.

Figure 1: Johann Wilhelm Schwedler (1823-1894)
Source: Zeitschrift für Bauwesen, 1895.

In 1863, Schwedler completed the dome above the gas container in Holzmarktstrasse 28, Berlin. He was the first engineer for the transition to a spatially dome that should be remembered as "*Schwedler dome*" in the literature. The second is the roof of the municipal gasworks in the Fichtestraße, in Berlin-Kreuzberg, erected in 1875 (Fig. 2). The latter dome has a diameter of 54,9 m and a rise of 12,2 m with an iron consumption of 28,7 kg /m²! This was a remarkable performance that remained unsurpassed in his lifetime [2].

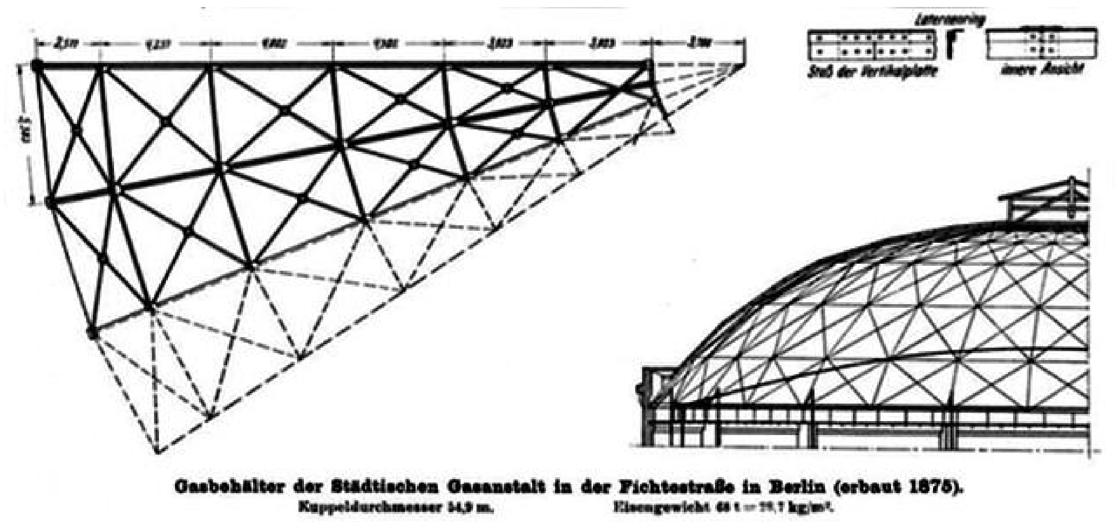

Figure 2: Schwedler dome; Source: [2]

Three years later, in 1866, Schwedler has published "*The Construction of the Dome Roof*" [3] in which he reported not only about his first truss domes, but also the theory of a simplified calculation method. The membrane stresses within the dome were projected on the longitudinal and latitudinal lines. The three-dimensional framework, which was statically indeterminate has particular system form due to the rotationally symmetrical configuration. The main structural elements of Schwedler dome were made of iron. The constructive concept of the rigid structure has two main elements: radially arched rafters and horizontal rings. They are stiffened by trusses in the form of tension crosswise and between rafters and fixed rings. The advantage of spatially cross-braced construction system was that the stability was ensured even with asymmetrical loads and each concentric ring constitutes a solid system that could be prefabricated.
The gained knowledge in this time, permitted to other engineers to design important domes. An example is the existing dome of Romanian Athenaeum in Bucharest, Romania. This building was inaugurated in 1888. One year later, the calculation notes of the dome were published by the Romanian engineer Elie Radu (1853-1931) [4]. He considered only one sector under the axisymmetrical loads.

3.2. The emerging mathematical solution

The equilibrium conditions for trusses with cyclic symmetry are studied by Hans Jacob Reissner. He published in 1908 the paper "*About trusses with cyclic symmetry*" [5]. Reissner considered series of trigonometric functions to analyze the stresses of the elements of a cyclic symmetrical structure.

One qualitative step forward in the analysis of cyclically symmetric domes was performed by Ludwig Mann [6]. He used of the forces method for the static calculation. The cyclicality of unknowns' coefficients of the system of equations is put into correlation with the properties of the n^{th} root of unity.

The two-story Schwedler dome has succeeded Kaufmann in 1921 [7], using the properties of cyclically symmetric systems to calculate the unknowns efforts from the equations system, by means of a cyclic determinant of order *n*, where *n* is the number of identical segments. Kaufmann divided the analysis problem into two independent specific tasks of analysis:

a) Calculation of the dome in a plane with bend rigid circuit ring;
b) Calculation of the dome in the meridian planes bending stiff continuous rafters.

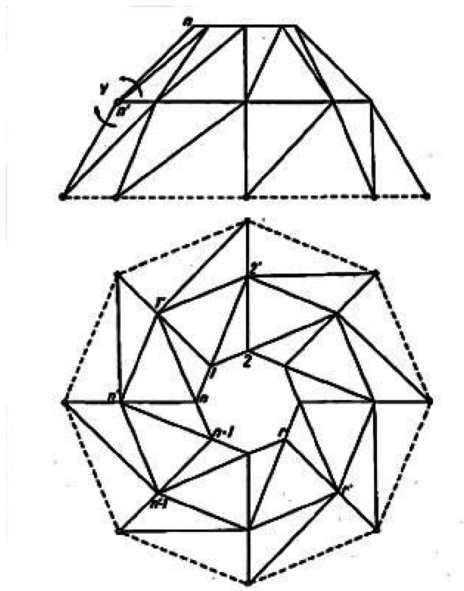

Figure 3: Schwedler dome with n=8 order of cyclicity.

The remarkable idea of symmetrical component three-phase systems has been published by Fortescue [8] in 1918. This was one of the first paper considering cyclic symmetry in the field of electrical engineering. Starting from the experiments, he carried out a mathematical model based on harmonic analysis. Later, this type of Fourier analysis will be one of the main instrument in the systematical approach of cyclical symmetric problems.

3.3. Richard Courant and the "birth" of finite element method (FEM)

The publication of Richard Courants's paper "*Variational Methods for Problems of Equilibrium and Vibration*" was a crucial contribution to the development of the FEM [9]. His work is unique.

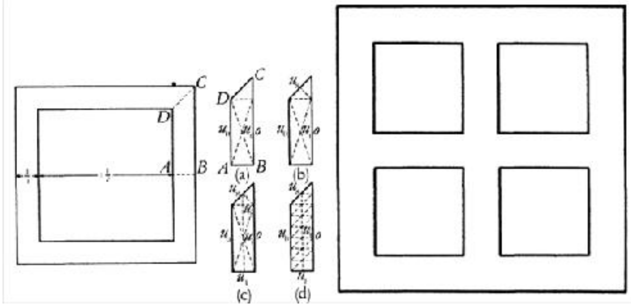

Figure 4

He gave us a body of purely mathematical work, and simultaneously offered the first example in numerical analysis that occur in applying the finite element method. In the appendix entitled "*Numerical treatment of the plane torsion problem for multiply-connected domains*" (Fig. 4), he referred also to the symmetry of his model to reduce the number of the unknowns:

> "We exploit the symmetry of the domain and the resulting symmetry of the solution; thus we may confine ourselves to considering only one-eighth of the domain *B**, namely the quadrangle *ABCD* " [10].

The Conference "*Finite Element Method-fifty years of the Courant element*" organized in Jyväskylä – Finland (1993), was dedicated to celebrate his prior recognition in FEM theory

4. DIGITAL COMPUTERS EPOCH

4.1. Matrix methods of structural analysis

The spearhead of the advanced theories in the structural analysis was carried out by the aircraft industries, in the sixties. The achievements of new technologies in modern digital computers have created the opportunity to elaborate extended and complex mathematical models of aircrafts. In this way, it was possible to obtain more precise results to design lighter and more resistant aircrafts. At that time, it was realized the potential of solving problems in continuum mechanics by using discrete elements.

By using matrix transformation methods it was clearly shown that most structural analysis methods could be categorized as either a force or a displacement method. The displacement method offers the advantage to work with a unique base system of the model and was adopted later like a standard in computing programmes for structures analysis.

The matrix methods of structural analysis developed for use on modern digital computers were systematized in parallel by Argyris in Germany and England [11-13], and Martin and Clough in America [14-15], based on the earlier theoretical papers of well-known scientists: Richard Courant (1943) [9], Walter Ritz (1908) [16], and J. W. S. Rayleigh (1870).

A different approach to the solution of continuum mechanics problems was realized by discrete element idealization. Therefore, analysis models for both continuous structures and frame structures were modeled as a system of elements interconnected at joints or nodes. Ray William **Clough** coined the terminology **finite element method** (FEM) of this new approach, in 1957.

4.2. Techniques and methods to exploit the symmetry

We consider symmetric structures subjected to a general loading. From the beginning of the structural analysis, the exploitation of bilateral or reflection symmetry is well known in structural mechanics by the use of "symmetry and anti-symmetry" techniques [17].

A different class of symmetry is the rotational or cyclic symmetry. Examples are the large radio telescope or space dome structures which can possess high degree of rotational symmetry.

This type of physical symmetric entity can be exploit by **discrete Fourier transform methods**. The first Fourier approach was developed by Fortescue [8] for the analysis of polyphase electrical circuits, as we shown. Renton [18] was the first in application discrete Fourier transformation methods to structural analysis. The "*General theory of cyclically symmetric frames*", is done by Hussey (1966-67) [19-20] in order to investigate the buckling under cyclically symmetric loading. Later, Thomas (1979) have deduced exact methods for solving eigenvalue problems and have demonstrate that any forced vibrations can be decomposed into independent rotating components [21]. The wave propagation in periodic structure was the subject in earlier papers of Mead in 1970 [22], and Orris and Petyt in 1974 [23].

In this type of analysis, the governing matrix (stiffness, mass, etc.) has a circulant or block-circulant form. A circulant matrix is a special matrix where each row vector is rotated one element to the right relative to the preceding row vector. An example of a circulant matrix is:

$$\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \\ 4 & 5 & 1 & 2 & 3 \\ 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix} \tag{1}$$

The theory of circulant matrices has been presented by Pipes "*Circulant matrices and the theory of symmetrical components*" [24] in 1966. Davis has published in 1979 his book "*Circulant matrices*" [25]. Characteristic for a mathematician fallen in love with "symmetry", he adopted the motto: "*What is circular is eternal*; *What is eternal is circular*" expressing in this way the magic of the results from his work.
Several problems of physics involve circulant matrices and block-circulant matrices. A pre-requisite of the Fourier method is that the coordinate system for the load and displacement vectors must have the same rotational symmetry as the structure.
By matrix transformations using the Fourier matrix, the governing matrix is reduced to a block-diagonalise form. This is the way to split the large system in sub-systems of equations. The characteristic of this type of analysis: every (n x n) circulant matrix has the same eigenvectors, and these are given by the columns of the Fourier matrix **F** [26]:

$$\mathbf{F} = \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & . & 1 \\ 1 & w & w^2 & . & w^{n-1} \\ 1 & w^2 & w^4 & . & w^{2(n-1)} \\ . & . & . & . & . \\ 1 & w^{n-1} & w^{2(n-1)} & . & w^{(n-1)^2} \end{bmatrix} \quad \text{where } w = e^{2®i/n} = \sqrt[n]{1} \qquad (2)$$

Kangwai, Guest and Peliegrino have presented that the Fourier method is a special case of the group representation theory method [26]. The same remark we can find in the works of Bossavit [27-30]. Dinkevici has considered in 1972, the method of spectral analysis, essentially identical to the discrete Fourier method [31].

4.3. Group theory and group representation theory

The best way to describe the full symmetry of any entity is the group theory. The properties of a symmetric structures are valorized at maximum by the group representation theory.
"The mathematical language of symmetry is group theory, and group representation theory is the vehicle for exploiting symmetry in linear problems" have stated Healey and Treacy in 1991 [32]. Another early example is "*Group Theory and its Applications to Physical Problems*" by Hammermesh (1962) [33]. The applications of group theory was studied by Willard Miller in "*Symmetry Groups and Their Applications*" [34], but the publications of Zloković in 1973 and respectively in 1989, are considered to be the first application of group representation theory to structural analysis [35-36]. A good reference (1976) is the doctoral thesis of Fässler "*Application of group theory to the method of finite elements for solving boundary value problems*" [37].
Nowadays, we can observe the increasingly number of articles using symmetry group operations. This is doubtless only the consequence of the universal feature of this method applied by researchers working in advanced new technologies domain. To have an idea of other diversity of domains (molecular vibrations, biological anisotropic hyperelastic materials, spectroscopy of crystals, crystallographic group theory in crystal chemistry, application of point-group symmetries in chemistry and physics), we can quote some relative recent articles [38-42].

4.4. FEM packages and cyclic symmetry

The Fourier approach to be applied for cyclically symmetric structure has been implemented first in the general purpose finite element package NASTRAN, in 1974 [43]. An example of a study in order to reduce computational effort is the paper "*Skyline solver for the static analysis of cyclic symmetric structures*" [44]. The procedure of Skyline has an efficient node numbering result and consequently, the time of processing data is significantly reduced. New techniques to reduce the hardware resources was the substructure method. The programming by means of a basic substructure for the cyclic symmetric analysis also improved the performances in the harmonic (Fourier) analysis [46].
The analysis of cyclically symmetric structures has become a common feature of FEM packages, e.g. ANSYS [47] and ABAQUS [48].
We can not omit the particular case when we consider a structure with a certain type of symmetry subjected to a loading with the same symmetry. The analysis is resume only of one identical substructure: Zienkiewicz in 1972 [49] and Glockner in 1973 [50].

5. INSTEAD OF CONCLUSIONS

The cyclic symmetry remains on-going subject in the theoretical and practical field of scientists. Even in this small field, there is already an enormous amount of work to be done to unearth inestimable treasures. The basic idea is to obtain a reduced problem before analyzing, to construct a problem having fewer unknowns than the original system.

Symmetry and group theory are important tools in analyzing physics and mechanics problems. The exploitation of symmetry via group invariance also yields an efficient computational approach to many problems in mechanical engineering: static, dynamic, stability, wave propagation, etc. Some comments of famous scientists reveal us the fascination of symmetry.

Hermann Klaus Hugo Weyl (1885 - 1955) concludes in his book "*Symmetry*" [51]:

> "Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect.
> I hope I have not completely failed in giving you an indication of its many ramifications, and in leading you up the ladder from intuitive concepts to abstract ideas."

Ian Nicholas Stewart (born 24 September 1945) pleaded for the beauty of mathematics in his book "*Why Beauty is Truth: A History of Symmetry*" [52]:

> "The implications of symmetry for physics, indeed for the whole of science, are still relatively unexplored. There is much that we do not yet understand. But we do understand that symmetry groups are our path through the wilderness-at least until a still more powerful concept (perhaps already waiting in some obscure thesis) comes along.
> In physics, beauty does not automatically ensure truth, but it helps.
> In mathematics, beauty must be true-because anything false is ugly."

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