LAPLACE OPERATOR ANALYSIS IN TRANSVERSE WAVES

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Abstract: *By studying and interpreting elastic waves, both at the theoretical level and at the level of mathematical modeling, one can observe the necessity of introducing the D`Alembert operator thus, the fundamental equation of the wave becomes better defined. Regarding the applicability of the mathematical model within the transverse elastic wave we can observe the equation deduced by Isaac Newton,the "vibrating string equation" , known as the keystone of the study of elastic waves.*
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1. INTRODUCTION

The notion of wave appeared in close connection with the concept of oscillation [3, 6]. In general, oscillation means a disturbance of a system that has a certain periodicity of a given physical size, which locally characterizes the state of a physical system.
Experience shows us that, in a material physical environment, any disturbance, regardless of its nature, propagates oscillatory "from close to close" through continuity, with a finite well-defined speed. The term wave is attributed to the space-time picture of a physical size whose perturbation propagates in a given environment.
Thus, according to the nature of the induced disturbance, we can distinguish the following fundamental categories from where [3, 6]:

a) Elastic waves, generated by the local mechanical disturbances of a physical system. They have material support that ensures propagation, the notion of substance. Thus, where elastic can exist in solid, liquid and gaseous environments, but they cannot exist in vacuum or in ether.

b) Electromagnetic waves, generated by electromagnetic disturbances (E.M.). These where they have the support of propagation the electromagnetic field (a special form of existence of matter) and they propagate in any space occupied by the substance or not, or the eternal ether.

c) Magnetohydrodynamic waves, present in plasma matter and which have as source complex plasma disturbances having both mechanical and electromagnetic characteristics.

d) Thermal waves, generated by thermal disturbances of a well-defined system.

e) Where "*Broglie*", these where they are associated with the microparticles whose movement they describe. Their existence is conditioned by the presence of a moving microparticle.

Due to this wide variety of system disturbances, a large number of physical sizes can have local variations in the environments in which a wave propagates. Among these sizes we must remember; displacement, speed, density, pressure, temperature, electric field intensity, magnetic field intensity, and others [4, 8].
For each case, a theoretical model used in the simulation field can be constructed, describing the propagation of a disturbance.
Once deduced and implicitly demonstrated the mathematical equations of propagation of the perturbations of various sizes of "*waves*", it can be seen that all waves have the same general mathematical form and depend in the same way on the nature of the environment in which they propagate. For these reasons, the general wave theory emerged as a purely formal theory, applicable to any type of wave. Universally accepted below we will note the size disturbed by the value ”Ψ”. Its value depends on both position and time.
Thereby, the function $\Psi = y(x,y,z,t)$ and it will be called "*wave function*". As the size Ψ reprezintă un scalar, vector sau tensor vom avea respectiv unde scalare, vectoriale or tensoriale represents a scalar, vector or tensor we will have respectively scalar, vector or tensor waves.
The vector waves can be transverse or longitudinal as the direction of oscillation of the perturbed size is perpendicular to the direction of propagation of the wave, respectively coinciding with it.

Thereby, an environment is homogeneous if its physical properties are the same at any point, that is, they are independent of the position coordinates (x, y, z). Otherwise, (corollary) the environment is defined as inhomogeneous.

Thereby, the environments in which a wave propagates impose certain characteristics of propagation. For this reason, it is necessary to characterize them. Thus, an environment is homogeneous if its physical properties are the same at any point, that is, they are independent of the position coordinates (x, y, z). Otherwise, (corollary) the environment is defined as inhomogeneous.

Averages are isotropic if their physical properties are the same for any direction in which they are measured. If they depend on the direction, the media are said to be anisotropic.

The environments in which wave propagation is done without entropy generation are called conservative environments and if the propagation process is accompanied by entropy generation then the environment is dissipative.

An environment is dispersive when the propagation speed of the wave depends on its frequency, in non-dispersive environments the propagation speed is independent of frequency.

An environment is called linear if the wave resulting from the superposition of several waves of wave functions Ψ_i, with i = 1 ... n, is described by the fundamental wave function (1):

$$\Psi(x.y.z.t.)= \sum_i \Psi i\ (x.y.z.t.) \qquad (1)$$

Otherwise the environment is nonlinear, in the sum defined in equation (1) and the higher powers of the functions and also appear.

A linear, homogeneous, isotropic, conservative and non-dispersive environment is called an ideal environment.

In the study of general wave theory we will stochastically estimate the propagation of waves in ideal environments, establishing the fundamental characteristics of the wave phenomena.

Where we deem it necessary, we will prove and study the dangers of propagating a wave in environments that, by certain properties, depart from the ideal environment.

2. THE WAVE EQUATION

In order to formulate the general wave equation let us consider an ideal one-dimensional "*ether*" environment such as, e.g. an ideal long continuous chord, figure 1.

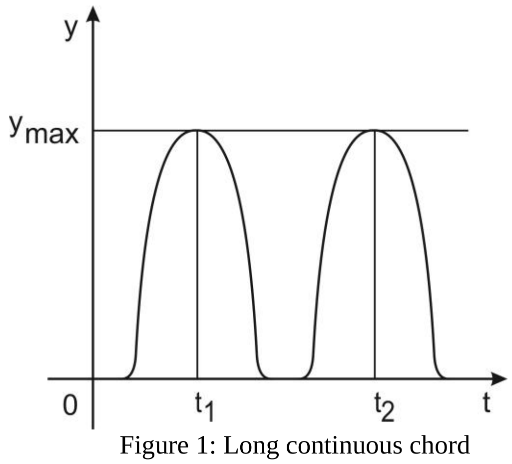

Figure 1: Long continuous chord

Let's assume that at the moment t_1 by an action before and outside the system under study, a disturbance of a magnitude appears y_{max} described by the function (2)

$$\Psi = \Psi_{x=0}(t_1) \qquad (2)$$

Experience shows that this disturbance propagates in the environment considered as a study so that at the moment t_2 the disturbance went through the unit of time $\Delta t = t_2 - t_1$.

If the environment under study is considered ideal, the shape of the disturbance will not change, it will have the same characteristics and its propagation will be at a constant speed. In these conditions we will have (3):

$$\Psi_{x=0}(t_1) = \Psi_x(t_2) \qquad (3)$$

Noting further with v as the speed at which the disturbance propagates in the studied environment, we observe that between the moments t_1 and t_2, with the specification that ($t_2 > t_1$) the following relationship will exist (4):

$$t_2 = t_1 - \frac{x}{v} \qquad (4)$$

Substituting t_2 in the equation (4), we get the equation (5):

$$\Psi_x(t)_1 = \Psi_{x=0}\left(t_2 - \frac{x}{v}\right) \quad (5)$$

Meaning, we'll get the equation (6):

$$\Psi(x,t) = \Psi_{x=0}\left(t_2 - \frac{x}{v}\right) \quad (6)$$

If the disturbance in point t_1 is a harmonic oscillation, the function$\Psi_{x=0}(t)_1$ it will take shape (7):

$$\Psi(t) = Ae^{i\omega t^2} \quad (7)$$

and the oscillation in the point t_2 will be described by the function (8):

$$\Psi(x,t) = Ae^{i\omega\left(t-\frac{x}{v}\right)} \quad (8)$$

We will note the value"k" the ratio between the pulsation ω of oscillation and speed v with which this propagates in the respective environment (9):

$$k = \frac{\omega}{v} \quad (9)$$

This ratio is even the module of the wave vector whose direction indicates the direction in which the wave propagates. How $\omega = \frac{2\pi}{T}$, this equation (9) becomes (10):

$$k = \frac{2\pi}{vT} = \frac{2\pi}{\lambda} \quad (10)$$

where we observe (11):

$$\lambda = vT \quad (11)$$

Thereby, λ represents the wavelength. This is a characteristic wave size and represents the distance traveled by a wave in a period Δt. Thus, the wave function (8) gets (12):

$$\Psi(x,t) = Ae^{i\omega(\omega t - kx)} \quad (12)$$

We mention that the argument of the wave function Ψ (x,t), respectively ω(x,t) = ωt - kx, represents the phase of the wave, and the surface on which the phase is constant is called the wavefront. The wave function (12) describes a plane harmonic wave which, moving along the *Ox* axis in the sense of positive *x*, it is a progressive wave.

Further defining the wave vector$\vec{k}$ as it is (13):

$$\vec{k} = \frac{\omega}{v}\,\vec{1}_k \quad (13)$$

where $\vec{1}_k$ is the direction of the propagation direction and considering the environment in which the three-dimensional propagation takes place, the wave function will be written (14):

$$\Psi(\vec{r},t) = Ae^{i(\omega t - \vec{k}\,\vec{r})} \quad (14)$$

We will see below that the wave function (14) satisfies a differential equation of the second order, linear and homogeneous, its explicit form resulting from simple operations.

To do this, let's calculate the second order derivatives of the function Ψ in relation to *x.y.z.* so (15):

$$\frac{\vartheta\Psi}{\vartheta x} = -ik_x\Psi, \quad \frac{\vartheta^2\Psi}{\vartheta x^2} = -k_x^2\Psi$$
$$\frac{\vartheta\Psi}{\vartheta y} = -ik_y\Psi, \quad \frac{\vartheta^2\Psi}{\vartheta y^2} = -k_y^2\Psi$$
$$\frac{\vartheta\Psi}{\vartheta z} = -ik_z\Psi, \quad \frac{\vartheta^2\Psi}{\vartheta z^2} = -k_z^2\Psi \quad (15)$$

Summarizing the partial derivatives of the second order and taking into account the fact that $k_x^2 + k_y^2 + k_z^2 = k^2$, we will get the equation (16):

$$\frac{\vartheta^2\Psi}{\vartheta x^2} + \frac{\vartheta^2\Psi}{\vartheta y^2} + \frac{\vartheta^2\Psi}{\vartheta z^2} = -k^2\Psi \quad (16)$$

an equation which can also be written in the form (17)

$$\nabla^2\Psi = -k^2\Psi \quad (17)$$

where (18):

$$\nabla^2 = \frac{\vartheta^2}{\vartheta x^2} + \frac{\vartheta^2}{\vartheta y^2} + \frac{\vartheta^2}{\vartheta z^2} \equiv \Delta \quad (18)$$

is defined as the Laplace operator.

If we continue to calculate the partial derivative of the second order of the function Ψ relative to time , we will get (19):

$$\frac{\vartheta\Psi}{\vartheta t} = i\omega\Psi, \ \frac{\vartheta^2\Psi}{\vartheta t^2} = -\omega^2\Psi \quad (19)$$

By combining the above results, the equation results immediately (20):

$$\nabla^2\Psi - \frac{1}{v^2}\frac{\vartheta^2\Psi}{\vartheta t^2} = 0 \quad (20)$$

Introducing the D`Alembert [3, 7] operator we will have (21):

$$\Box \equiv \nabla^2 - \frac{1}{v^2}\frac{\vartheta^2}{\vartheta t^2} \quad (21)$$

this equation takes on a much more concise form, so we identify (22):

$$\Box\Psi = 0 \quad (22)$$

Equations (21) or (22) represent the general equation of the waves in absolutely ideal-etheric environments.

3. TRANSVERSAL ELASTIC WAVE

The D'Alembert equation [3, 7] describes the propagation of any type of wave in an ideal environment. We will be able to find such equations, establishing the concrete expression of the propagation speed for the transverse elastic waves, which can exist only in solid bodies, as well as for the longitudinal elastic waves that propagate in both solids and fluids.

Transverse elastic waves appear, (e.g. along an elastic cord fastened to both ends). By removing a small portion of the string from the equilibrium position [3, 5], a local disturbance occurs which due to the interaction forces between the disturbed point and the neighboring points propagates along the string.

Consider, further, such a chord, homogeneous, of mass (*m*), whose points p_1 and p_2, following the induced excitation perform oscillations in the vertical plane. Let us note further with $\vec{F}$ the tension in the rope, the same along the entire length of the given string, figure 2. If the deformations are small enough, the angles α and α', with α' = α + dα, are also small so that sin α ≅ tg α ≅ α, and cos α ≅ cos α` ≅ 1. Also, the slope of the string element *K* is small, respectively (23):

$$K = \frac{\vartheta \Psi}{\vartheta x} = \frac{F_y}{F} \cong \frac{F`_y}{F} \ll 1 \quad (23)$$

By graphically exemplifying the above equation we obtain figure 2.

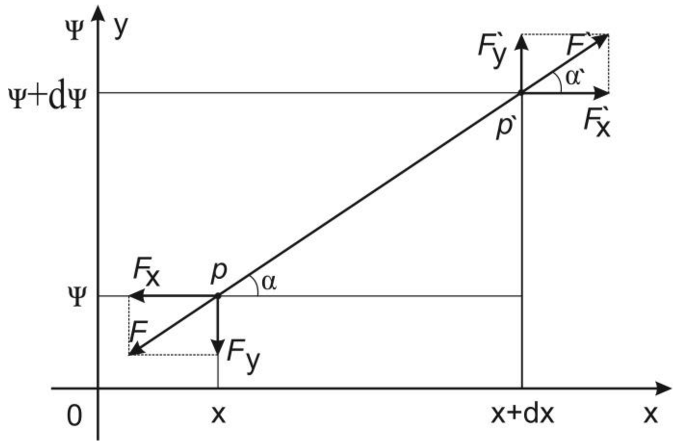

Figure 2: Transverse elastic wave

As obviously (24):

$$F_x = \sqrt{F^2 - F_y^2} \quad (24)$$

for small deformations, we will have (25):

$$F_x = F\sqrt{1-K^2} \cong F \quad (25)$$

Similarly, if the relation (25) for the component is considered $F_x^`$ we will get the same value (26):

$$F_x = \sqrt{F^2 - F_y^2} = F\sqrt{1-K^2} \cong F \quad (26)$$

The last two relationships show us that the result of the components of the tension F along the axis O_x, namely (28):

$$dF_x = F_x^` - F_x \quad (28)$$

is zero, so the introduced transverse disturbance does not cause oscillations according to the direction O_x. The result of the tension forces acting on the rope portion is a force dF_y whose size will be (29):

$$dF_y = F_y(x + dx) - F_y(x) = F[K(x + dx) - K(x)] \quad (29)$$

Developing in series K(x + dx) and keeping only the first two terms we will get (30):

$$dF_y = \left[K(x) + \frac{\vartheta K}{\vartheta x}dx - K(x)\right] = F\frac{\vartheta K}{\vartheta x}dx \quad (30)$$

Taking into account his expression *K* (23), the equation (30) becomes (31):

$$dF_y = F\frac{\vartheta^2 \Psi}{\vartheta x^2}dx \quad (31)$$

The action of force dF_y on the mass string element (*dm*) causes an acceleration a_y of the following form (32):

$$a_y = \frac{\vartheta^2 \Psi}{\vartheta x^2}dx \quad (32)$$

as it results from the fundamental equation of dynamics, namely (33):

$$dF_y = (dm)a_y \quad (33)$$

How ”*dm=ρSdl”*, where ρ represents the mass density, S represents the string section, and $dl \cong dx$. Taking into account equation (32), equation (33) becomes (34):

$$F\frac{\vartheta^2\Psi}{\vartheta x^2}dx = \rho S\frac{\vartheta^2\Psi}{\vartheta t^2}dx \quad (34)$$

From here, the differential equation of the transverse elastic waves produced in a vibrating string [1] immediately results (35):

$$\frac{\vartheta^2\Psi}{\vartheta x^2} - \frac{1}{v^2}\frac{\vartheta^2\Psi}{\vartheta t^2} = 0 \quad (35)$$

where we have (36):

$$v = \sqrt{\frac{F}{\rho S}} = \sqrt{\frac{F}{m_0}} = \sqrt{\frac{\tau}{\rho}} \quad (36)$$

Where v is the speed at which the transverse wave propagates in the vibrating chord [1, 2] of mass density (*ρ)*, respectively of linear mass density (m_0) where the mechanical tension is”*F*”, respectively the unitary effort will have a value of (37):

$$\tau = \frac{F}{S} \quad (37)$$

Equation (37) was deduced from Isaac Newton and is respectively known as the "*vibrating string equation*".

4. CONCLUSION

We can conclude by the fact that the study of the waves and their characteristics, using d’Alembert operator, is the keystone of the wave dynamics through different propagation media, thus allowing the identification and solving, at the theoretical level, of the various errors and ambiguities that can be made, and can slip into different computer-assisted computing systems, which can lead to real-time rectification and recalibration of mathematical wave models as well as all their related applications.

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