A STUDY OF MINDLIN PLATE FINITE ELEMENTS

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Abstract: *Between finite element applications, bending plate elements are the most frequently used in structural analysis. However the advanced topics which are at the basis of the most part of these elements cannot be fully covered by the engineering educational process. This paper contains a study of Mindlin plate finite elements in order to find or even reformulate elements, such that their presentation becomes as simple as possible. Issues like, parametric functions, reduced and selective integration, bubble functions or linked interpolation are avoided, but increased computational efforts are accepted if they simplify the formulation. A displacement formulation triangle in metric coordinates was found, whith correct rank, reduced level of locking, and a relative simple formulation. The presented numerical examples show an accuracy of the results, which is comparable with known elements from available commercial structural analysis software.*
Keywords: *finite elements, Mindlin plate triangle, displacement formulation*

1. INTRODUCTION

The development of the six node Mindlin plate triangular finite element presented in this paper has an educational purpose. The ecuations involved in the formulation of the element are simple. The element can be understood with the basic plate bending theory and without advanced finite element knowledge. The element passes the constant bending and shear patch tests and has correct rank. The presented numerical examples show good performances.

2. MINDLIN PLATE ECUATIONS

For the plate of figure 1. the displacements of a point *P(x,y,z)* are:

$$u = z\varphi_x(x,y),\quad v = z\varphi_y(x,y),\quad w = w(x,y). \tag{1}$$

The plane xy of the reference system is the median plane of the plate. φ_x and φ_y are the rotations of the normal in the *xz* and respectively *yz* plane.

$$\varphi_x = \partial u/\partial x = u_{,x};\quad \varphi_y = \partial v/\partial y = u_{,y}. \tag{2}$$

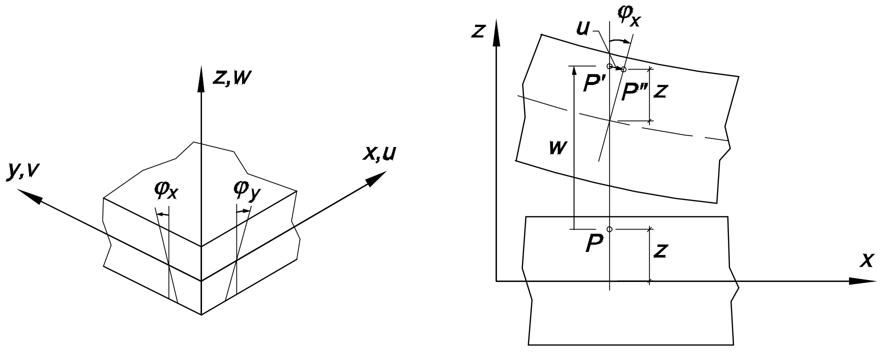

Figure 1: Displacements and rotations

The strains are pure bending strains:

$$\varepsilon_x = u_{,x} = z\varphi_{x,x},$$
$$\varepsilon_y = v_{,y} = z\varphi_{y,y}, \qquad \textbf{(3)}$$
$$\gamma_{xy} = u_{,y} + v_{,x} = z(\varphi_{x,y} + \varphi_{y,x}),$$

and transversal shear strains:

$$\gamma_{xz} = u_{,z} + w_{,x} = \varphi_x + w_{,x},$$
$$\gamma_{yz} = v_{,z} + w_{,y} = \varphi_y + w_{,y}. \qquad \textbf{(4)}$$

For an izothrop linear elastic material the tensions are:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & (1-\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & (1-\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}. \qquad \textbf{(5)}$$

The stress resultants used in bending plate applications are:

$$m_x = \int_{-t/2}^{t/2} z\sigma_x dz, \quad m_y = \int_{-t/2}^{t/2} z\sigma_y dz, \quad m_{xy} = \int_{-t/2}^{t/2} z\tau_{xy} dz,$$
$$t_x = \int_{-t/2}^{t/2} \tau_{xz} dz, \quad t_y = \int_{-t/2}^{t/2} \tau_{yz} dz. \qquad \textbf{(6)}$$

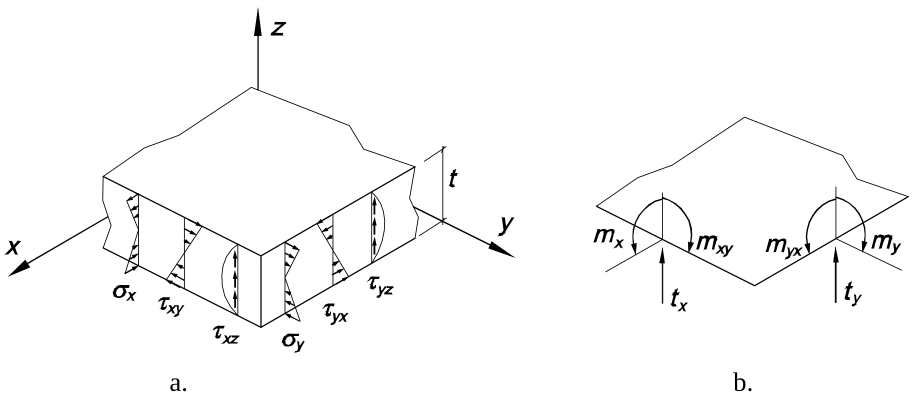

Figure 2: a. Tensions, b. Stress resultants

Here m_x and m_y are bending moments, m_{xy}=m_{yx} is the torsional moment, t_x and t_y are the shear forces.

$$\sigma_x^{\min,\max} = \pm 6m_x / t^2, \quad \sigma_y^{\min,\max} = \pm 6m_y / t^2, \quad \tau_{xy}^{\min,\max} = \pm 6m_{xy} / t^2. \qquad \textbf{(7)}$$

The transversal shear tensions τ_{xz} and τ_{yz} usually are small. Their variation is quadratic on the thickness of the plate.

$$\tau_{xz}^{\max} = 1.5t_x / t, \quad \tau_{yz}^{\max} = 1.5t_y / t. \qquad \textbf{(8)}$$

From the equations (3)-(6), results:

$$\begin{Bmatrix} m_x \\ m_y \\ m_{xy} \\ t_x \\ t_y \end{Bmatrix} = \begin{bmatrix} D & \nu D & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 \\ 0 & 0 & (1-\nu)D/2 & 0 & 0 \\ 0 & 0 & 0 & kGt & 0 \\ 0 & 0 & 0 & 0 & kGt \end{bmatrix} \begin{Bmatrix} \varphi_{x,x} \\ \varphi_{y,y} \\ \varphi_{x,y} + \varphi_{y,x} \\ \varphi_x + w_{,x} \\ \varphi_y + w_{,y} \end{Bmatrix}, \qquad \textbf{(9)}$$

or: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$.

Here $D = \dfrac{Et^3}{12(1-\nu^2)}$, and k introduces the effect of nonuniform shear deformations on the thickness of the plate. For isothropic material k=5/6.

3. THE T6w3 ELEMENT

The element has six nodes. Nodes 4, 5, 6 are in the middle of the straight edges of the element. The coordinates of the nodes are: $\mathbf{x}_e = \begin{pmatrix} x_1 & x_2 & \cdots & x_6 \end{pmatrix}^T$ and $\mathbf{y}_e = \begin{pmatrix} y_1 & y_2 & \cdots & y_6 \end{pmatrix}^T$.

The displacements of the nodes are: $\mathbf{a} = \begin{Bmatrix} \mathbf{w} \\ \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y \end{Bmatrix} = \begin{pmatrix} w_1 & w_2 & \cdots & w_6 & \theta_{x1} & \theta_{x2} & \cdots & \theta_{x6} & \theta_{y1} & \theta_{y2} & \cdots & \theta_{y6} \end{pmatrix}^T$.

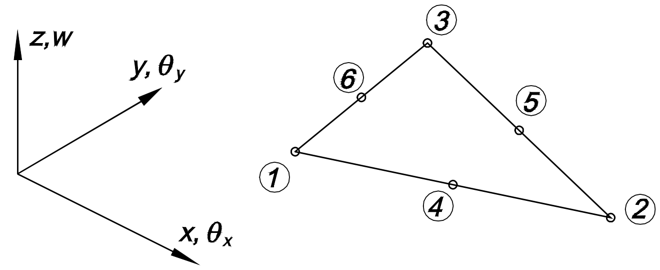

Figure 3: The T6w3 element

The internal displacement field is:

$$\mathbf{u} = \begin{Bmatrix} w \\ \varphi_x \\ \varphi_y \end{Bmatrix} = \begin{bmatrix} \mathbf{P}_3 & \mathbf{0} & \mathbf{0} & 0 & 0 \\ -\mathbf{P}_{3,x} & \mathbf{P}_1 & \mathbf{0} & 2xy & -y^2 \\ -\mathbf{P}_{3,y} & \mathbf{0} & \mathbf{P}_1 & -x^2 & 2xy \end{bmatrix} \cdot \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{18} \end{Bmatrix} = \mathbf{P}\boldsymbol{\alpha} \tag{10}$$

where $\mathbf{P}_3 = \begin{pmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 \end{pmatrix}$, $\mathbf{P}_{3,x} = \begin{pmatrix} 0 & 1 & 0 & 2x & y & 0 & 3x^2 & 2xy & y^2 & 0 \end{pmatrix}$,
$\mathbf{P}_{3,y} = \begin{pmatrix} 0 & 0 & 1 & 0 & x & 2y & 0 & x^2 & 2xy & 3y^2 \end{pmatrix}$ and $\mathbf{P}_1 = \begin{pmatrix} 1 & x & y \end{pmatrix}$.

$\mathbf{P}_3$ contains the functions of a complete polynomial of degree three in x and y, $\mathbf{P}_{3,x}$ and $\mathbf{P}_{3,y}$ are its derivatives, which are incomplete degree two polynomials.

The rotations φ_x and φ_y are given by the derivatives of the transversal displacements, completed by linear transversal shear deformations described by six independent parameters α_{11}, ... , α_{16}. By adding the terms $\varphi_x = 2\alpha_{17}xy - \alpha_{18}y^2$ and $\varphi_y = -\alpha_{17}x^2 + 2\alpha_{18}xy$, the rotation fields become complete degree two polynomials.

The unknow parameters $\boldsymbol{\alpha}=(\alpha_1\ \alpha_2\ ...\ \alpha_{18})^T$ can be determined from the nodal displacements of the element.
The relations between the nodal rotations θ which are vectorial quantities and the internal rotations φ which are slopes are: $\theta_x = -\varphi_y$ and $\theta_y = \varphi_x$. The relation between the nodal displacements and the internal displacement field parameters α is:

$$\begin{Bmatrix} \mathbf{w} \\ \boldsymbol{\theta}_x \\ \boldsymbol{\theta}_y \end{Bmatrix} = \begin{bmatrix} \mathbf{P}_3(\mathbf{x}_e,\mathbf{y}e) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{P}_{3,y}(\mathbf{x}_e,\mathbf{y}e) & \mathbf{0} & -\mathbf{P}_1(\mathbf{x}_e,\mathbf{y}e) & \mathbf{x}_e^2 & -2\mathbf{x}_e\mathbf{y}_e \\ -\mathbf{P}_{3,x}(\mathbf{x}_e,\mathbf{y}e) & \mathbf{P}_1(\mathbf{x}_e,\mathbf{y}e) & \mathbf{0} & 2\mathbf{x}_e\mathbf{y}_e & -\mathbf{y}_e^2 \end{bmatrix} \begin{Bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \vdots \\ \boldsymbol{\alpha}_{18} \end{Bmatrix} \tag{11}$$

Or: $\mathbf{a} = \mathbf{C}\boldsymbol{\alpha}$.

Results $\boldsymbol{\alpha} = \mathbf{C}^{-1}\mathbf{a}$.
Using the above relation, the displacement field becomes:

$$\mathbf{u} = \mathbf{P}\mathbf{C}^{-1}\mathbf{a} = \mathbf{N}\mathbf{a}\ . \tag{12}$$

Here $\mathbf{N}$ contains the displacement interpolation functions. The transversal displacements depend both on the nodal displacements and rotations. It can be shown that the displacements on the common edge of two neighbouring elements are compatible.
The displacement derived strains are:

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varphi_{x,x} \\ \varphi_{y,y} \\ \varphi_{x,y} + \varphi_{y,x} \\ \varphi_x + w_{,x} \\ \varphi_y + w_{,y} \end{Bmatrix} = \begin{bmatrix} -\mathbf{P}_{3,xx} & \mathbf{P}_{1,x} & \mathbf{0} & 2y & 0 \\ -\mathbf{P}_{3,yy} & \mathbf{0} & \mathbf{P}_{1,y} & 0 & 2x \\ -2\mathbf{P}_{3,xy} & \mathbf{P}_{1,y} & \mathbf{P}_{1,x} & 0 & 0 \\ \mathbf{0} & \mathbf{P}_1 & \mathbf{0} & 2xy & -y^2 \\ \mathbf{0} & \mathbf{0} & \mathbf{P}_1 & -x^2 & 2xy \end{bmatrix} \mathbf{C}^{-1}\mathbf{a} = \mathbf{B}\mathbf{a} \quad , \tag{13}$$

where $\mathbf{P}_{3,xx} = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 & 0 \end{pmatrix}$,

$\mathbf{P}_{3,yy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2x & 6y \end{pmatrix}$ and

$\mathbf{P}_{3,xy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 \end{pmatrix}$.

The equilibrium of an element can be expressed as:

$$\mathbf{ka} = \mathbf{f} \;. \quad \textbf{(14)}$$

Where

$$\mathbf{k} = \int_{V_e} \mathbf{B}^T \mathbf{D} \mathbf{B} dv \;, \quad \textbf{(15)}$$

is the stiffness matrix of the element and

$$\mathbf{f} = \int_{A_e} \mathbf{N}^T \mathbf{q} dA \quad \textbf{(16)}$$

are the nodal forces resulted from the loads **q** distributed on the A_e surface of the element.

Integral (15) on the volume V_e of the element is computed by the midpoint quadrature rule which is exact for quadratic integrands and reduces the shear locking effect introduced by the $\varphi_x = 2\alpha_{17}xy - \alpha_{18}y^2$ and $\varphi_y = -\alpha_{17}x^2 + 2\alpha_{18}xy$, rotation terms. For uniform transversal load q, (16) becomes: $\mathbf{f} = qA_e/3*(0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)^T$.

4. NUMERICAL EXAMPLES

4.1. Patch test

The T6w3 element passes the constant bending and shear patch tests given by MacNeal and Harder [2].

4.2. Cantilever beam

A cantilever beam modeled by two t6w3 elements fully clamped at one end is subjected to three load cases as shown in figore 4. To avoid the anticlastic curvature effect, Poisson ratio of the material is taken to be zero.

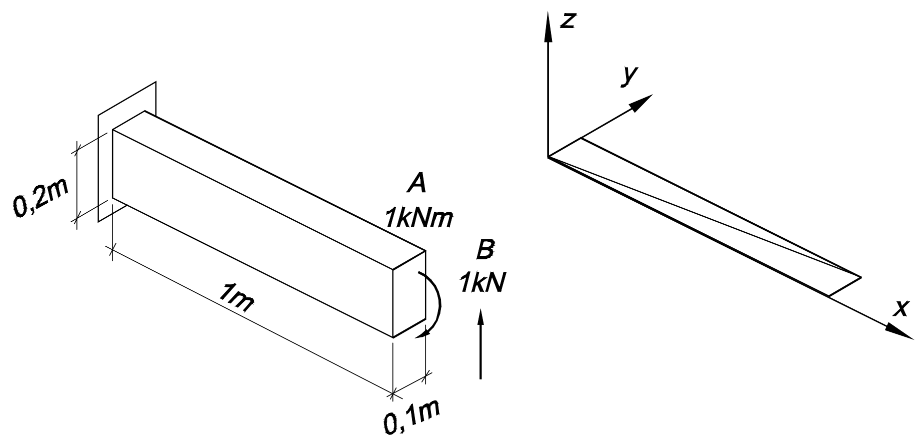

Figure 4: Cantilever beam - $E = 10^7$ (kN/m^2); $\nu = 0$.

Table 1: Maximum tip displacements (mm)

	Case A Concentrated Moment	Case B Concentrated Force	Case C Uniform unit transversal load
Beam theory	-0.075	0.0512	0.01935
T6w3 element	-0.075	0.0512	0.01935

4.3. Uniformly loaded square plate

A square plate of side L is considered. A quadrant of the plate is modelled by 2x2, 4x4, ... , 32x32 meshes. In the table 2 centre transversal displacements are presented for hard simply support on the sides (w=0, θ_n=0) and two L/t ratios. Table 3 contains centre transversal displacements for the clamped case (w=0, θ_t=0). Although the element is not completely free of shear locking, the results are good, especially for thick plates. In the figures 6, 7 and 8 bending moments, torsional moments and shear forces are presented for a fully calmped thick plate (L/t=10, w=0, θ_t=0, θ_n=0).

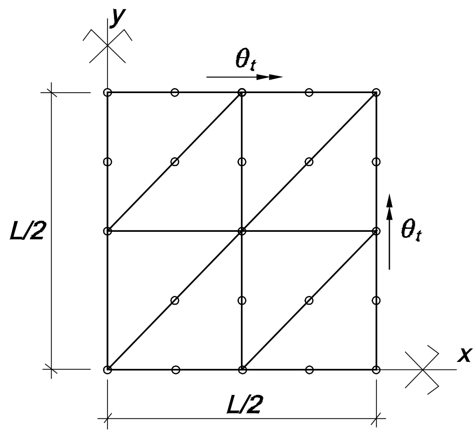

Figure 5: The model of a quadrant of a square plate (N=2); E=10.92, ν=0.3, L=10; q=1

Table 2: Centre displacements of the uniformly loaded simply supported plate

Mesh, N	L/t=10, w·10		L/t=1000, $w \cdot 10^{-7}$	
	T6w3	[1]	T6w3	[1]
2	4.2796	4.2626	4.0756	4.0389
4	4.2734	4.2620	4.0623	4.0607
8	4.2728	4.2727	4.0624	4.0637
16	4.2728	4.2728	4.0624	4.0643
32	4.2728	4.2728	4.0624	4.0644
Series		4.2728		4.0624

Table 3: Centre displacements of the uniformly loaded clamped plate

Mesh, N	L/t=10, w·10		L/t=1000, $w \cdot 10^{-7}$	
	T6w3	[1]	T6w3	[1]
2	1.5170	1.4211	0.9863	1.1469
4	1.5045	1.4858	1.2504	1.2362
8	1.5044	1.4997	1.2648	1.2583
16	1.5046	1.5034	1.2653	1.2637
32	1.5046	1.5043	1.2653	1.2646
Series		1.499		1.2653

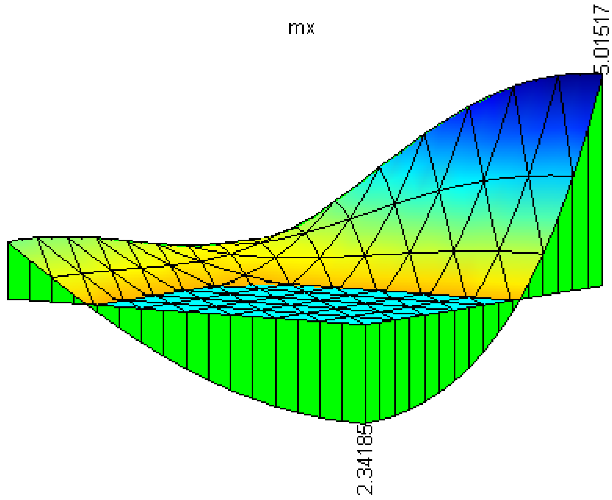

Figure 6: Bending moments for the uniformly loaded clamped square plate
Reference values: 2.328, 5.021

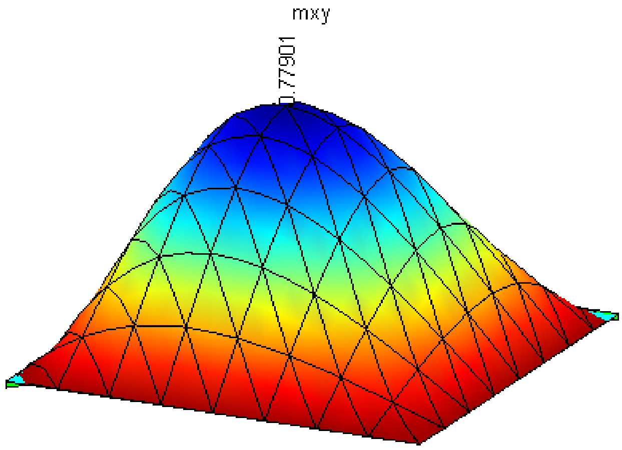

Figure 7: Torsional moments for the uniformly loaded clamped square plate
Reference value: 0.773

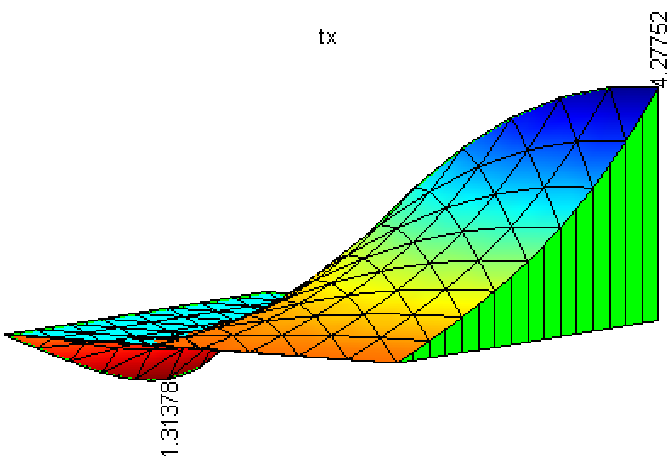

Figure 8: Shear forces for the uniformly loaded clamped square plate
Reference values: 1.327, 4.293

4. CONCLUSION

In this paper a six node Mindlin plate triangular finite element is presented. The internal displacements of the element are described by a complete cubic polynomial for the transversal displacements and quadratic rotations in metric coordinates. The element passes the constant bending and shear patch tests. Although the element is not completely free of shear locking, from numerical examples results, that it has comparable performances with the best known elements of its type.

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