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THE POWER SPECTRAL DENSITY OF THE STATIONARY RESPONSE OF DUFFING OSCILLATOR WITH NONLINEAR ELASTIC CHARACTERISTIC

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Abstract: *We present a method for estimating the power spectral density of the stationary response of oscillator with a nonlinear restoring force under external stochastic wide-band excitation. If a non-linear mathematical model of the system under consideration is adopted, together with a random process model of the excitation, then one is faced with the problem of predicting the system response. Non-mechanical system are completely linear, but linearization around the equilibrium position is acceptable in many cases-at least within some range of deformation. Linearization is important because linear functions are easier to deal with. Using linearization, one can estimate function values near known points. The statistical linearization technique can also tackle a wide variety of problems and also provides approximate information on the frequency domain characteristics of the stochastic response.*
Keywords: *nonlinear vibration, stationary response, spectral density function.*

1. SYSTEM MODEL

Consider the following oscillator with a nonlinear restoring force component, exposed to the simultaneous action of n forces $W_1(t), W_2(t), \ldots W_m(t)$, where $x_{W_1}(t), x_{W_2}(t), \ldots x_{W_m}(t)$ denote the effect of the forces on the system response, when the forces are applied separately. The ordinary differential equation of the motion can be written as

$$m\ddot{x}(t) + c\dot{x}(t) + g(x(t)) = W(t), \tag{1}$$

where m is the mass, c is the viscous damping coefficient, $W(t)$ is the external excitation signal with zero mean and $x(t)$ is the displacement response of the system. The Fourier Transform is a generalization of the Fourier series. Strictly speaking it applies to continuous and aperiodic functions, but the use of the impulse function allows the use of discrete signals. The set of conditions that guarantee the existence of the Fourier transform is the Dirichlet conditions, which may be expressed as: the signal $x(t)$ has a finite number of finite discontinuities, the signal $x(t)$ contains a finite number of maxima and minima and the signal $x(t)$ is absolutely integrable.
Dividing the equation by m, the equation of motion can be rewritten as:

$$\ddot{x}(t) + 2\xi p\dot{x}(t) + h(x(t)) = w(t) \tag{2}$$

where w(t) is a zero mean stationary Gaussian white noise excitation. We can always find a way to decompose the nonlinear restoring force to one linear component plus a nonlinear component

$$h(x) = p^2(x + \frac{1}{\psi}Q(x(t))), \tag{3}$$

where ψ is the nonlinear factor to control the type and degree of nonlinearity in the system. We consider in this article the nonlinear factor $Q(x)$ of the form $Q(x(t)) = a_n x^n(t) + a_{n-1}x^{n-1}(t) + \ldots + a_1 x(t) + a_0$. The equation of motion, in this case, can be rewritten as:

$$\ddot{x}(t) + 2\xi p\dot{x}(t) + p^2 x(t) + \frac{1}{\psi}p^2\left(a_n x^n(t) + a_{n-1}x^{n-1}(t) + \ldots + a_1 x(t) + a_0\right) = w(t), \tag{4}$$

where $\xi = \dfrac{c}{2pm}$.

Obtain

$$\ddot{x}(t) + 2\xi_e p_e\dot{x}(t) + p_e^2 x(t) = w(t), \tag{5}$$

where p_e is the undamped natural frequency and ξ_e is the critical damping factor. For the linear system $\xi_e = \frac{p}{p_e}\xi$. The difference between the nonlinear stiffness [1,2] and linear stiffness terms is

$$e = p^2[x(t) + \frac{1}{\psi}\left(a_n x^n(t) + a_{n-1}x^{n-1}(t) + ... + a_1 x(t) + a_0\right)] - p^2{}_e x(t) \tag{6}$$

The nonlinear factor ψ controls the type and degree of nonlinearity in the system. The value of p_e can be obtained [3,4] by minimizing the expectation of the square error

$$\frac{dE\{e^2\}}{dp_e{}^2} = 0 . \tag{7}$$

Because

$$E\{x^2(t)\} = \sigma_x{}^2 = \int_{-\infty}^{\infty} x^2(t)P(x(t))dx , \tag{8}$$

obtain

$$p_e{}^2 = p^2\left(1 + \psi\frac{E\left\{x(t)\left(a_n x^n(t) + a_{n-1}x^{n-1}(t) + ... + a_1 x(t) + a_0\right)\right\}}{\sigma_x{}^2}\right). \tag{9}$$

with $a_n, a_{n-1}, ..., a_0 > 0$.

The probability density function of the system, for a normal distribution is

$$P(x) = \frac{e^{\frac{-x^2(t)}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}}. \tag{10}$$

We have

$$E\left\{x(t)\left(a_n x^n(t) + a_{n-1}x^{n-1}(t) + ... + a_1 x(t) + a_0\right)\right\} =$$

$$= \int_{-\infty}^{\infty} x(t)\left(a_n x^n(t) + a_{n-1}x^{n-1}(t) + ... + a_1 x(t) + a_0\right)\cdot\frac{e^{\frac{-x^2(t)}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}}dx =$$

$$= \int_{-\infty}^{\infty} a_n x^{n+1}(t)\cdot\frac{e^{\frac{-x^2(t)}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}}dx + \int_{-\infty}^{\infty} a_{n-1}x^n(t)\cdot\frac{e^{\frac{-x^2(t)}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}}dx + ... + \tag{11}$$

$$+ \int_{-\infty}^{\infty} a_1 x^2(t)\cdot\frac{e^{\frac{-x^2(t)}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}}dx + \int_{-\infty}^{\infty} a_0 x(t)\cdot\frac{e^{\frac{-x^2(t)}{2\sigma_x^2}}}{\sigma_x\sqrt{2\pi}}dx$$

For $S_w(\omega) = S_0 = ct.$, the standard deviation of $x(t)$ is

$$\sigma^2{}_x = \frac{\pi S_0}{cp_e{}^2} = \frac{\pi S_0}{cp^2\left(1 + \psi\frac{E\left\{x(t)\left(a_n x^n(t) + a_{n-1}x^{n-1}(t) + ... + a_1 x(t) + a_0\right)\right\}}{\sigma_x{}^2}\right)} . \tag{12}$$

Using the Fourier transform [5,6] we obtain

$$\begin{cases} \bar{x}_1(\omega) = (p_e{}^2 - \omega^2 + 2\xi_e p_e \omega i) F_{W_1}(\omega) \\ \bar{x}_2(\omega) = (p_e{}^2 - \omega^2 + 2\xi_e p_e \omega i) F_{W_2}(\omega) \\ \text{...} \\ \bar{x}_m(\omega) = (p_e{}^2 - \omega^2 + 2\xi_e p_e \omega i) F_{W_m}(\omega) \end{cases} \tag{13}$$

where

$$\mathrm{F}\ (\dot{x}_k(t)) = i\omega \bar{x}_k(\omega),\ ,\ \mathrm{F}\ (W_k(t)) = F_{W_k}(\omega)\ ,\ k = 1,2,...,m.$$

As the complex conjugates of these expressions are

$$\begin{cases} \bar{x}^*{}_1(\omega) = H^*(\omega) F^*{}_{W_1}(\omega) \\ \bar{x}^*{}_2(\omega) = H^*(\omega) F^*{}_{W_2}(\omega) \\ \text{...} \\ \bar{x}^*{}_m(\omega) = H^*(\omega) F^*{}_{W_m}(\omega) \end{cases} \tag{14}$$

the searched quantity is obtained as

$$\left|\bar{x}(\omega)\right|^2 = [\bar{x}_1(\omega) + \bar{x}_2(\omega) + ...\bar{x}_m(\omega)][\bar{x}^*{}_1(\omega) + \bar{x}^*{}_2(\omega) + ...\bar{x}^*{}_m(\omega)] =$$

$$= \sum_{j=1}^{m} \bar{x}_j(\omega) \sum_{j=1}^{m} \bar{x}^*{}_j(\omega) = \left|H(\omega)\right|^2 [\left|F_{W_1}(\omega)\right|^2 + \left|F_{W_2}(\omega)\right|^2 ... \left|F_{Wn}(\omega)\right|^2 + \sum_{\substack{i=1 \\ j=1 \\ i<j}}^{m} F^*{}_{W_i}(\omega) F_{W_j}(\omega). \tag{15}$$

The mixed products correspond to the cross-spectral density functions, or simply cross spectral $S_{W_i W_j}(\omega),\ i = 1,...m,\ j = 1,...m,\ i \neq j.$

The frequency response function of the system is given by equation

$$\frac{1}{H(\omega)} = k_e{}^2 - m\omega^2 + c\omega i \tag{16}$$

or

$$\frac{1}{H(\omega)} = m\left[p^2\left(1 + \psi \frac{E\left\{x(t)\left(a_n x^n(t) + a_{n-1}x^{n-1}(t) + ... + a_1 x(t) + a_0\right)\right\}}{\sigma_x{}^2}\right) - \omega^2 + 2\xi p \omega i\right] \tag{17}$$

Because the cross spectral density function can be expressed in terms of the cross-correlation function

$$R_{W_i W_j}(\tau) = \lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} W_i(t) W_j(t+\tau) dt,\ i \neq j, \tag{18}$$

obtain for the cross spectral density function

$$S_{W_i W_j} = 2\int_{-\infty}^{\infty} R_{W_i W\ j}(\tau) e^{-i\omega t} dt = \int_{-\infty}^{\infty} \left(\lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} W_i(t) W_j(t+\tau) dt \right) e^{-i\omega t} dt \tag{19}$$

The cross spectral density function can be also written in the form

$$\begin{aligned} S_{W_r W_s}(\omega) = 2\int_{-\infty}^{\infty} \left(\lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} W_i(t) W_j(t+\tau) dt \right) \cos \omega\tau d\tau + \\ +2i\int_{-\infty}^{\infty} \left(\lim_{T \to \infty} \int_{-\frac{T}{2}}^{\frac{T}{2}} W_i(t) W_j(t+\tau) dt \right) \sin \omega\tau d\tau \end{aligned} \tag{20}$$

The autocorrelation of the process $W(t)$ can be evaluated as

$$R_W(\tau)=\lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}}[W_1(t)W_1(t+\tau)+W_1(t)W_2(t+\tau)+$$
$$+...+W_1(t)W_m(t+\tau)+W_2(t)W_1(t+\tau)+W_2(t)W_2(t+\tau)+ \qquad (21)$$
$$+...+W_2(t)W_m(t+\tau)]dt=\sum_{j=1}^{m}R_{W_j}(\tau)+\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}R_{W_iW_j}(\tau).$$

The variance [3] of the process $W(t)$ can be written as

$$\sigma^2{}_W=R_W(0)=\sum_{j=1}^{m}R_{W_j}(0)+\sum_{\substack{i=1\\j=1\\i\neq j}}^{m}R_{W_rW_s}(0). \qquad (22)$$

If the processes $W_1(t)\,,W_2(t),...W_m(t)$ are statistically independent then the variance is obtained as $\sigma^2{}_W=\sigma^2{}_{W_1}+\sigma^2{}_{W_2}+...\sigma^2{}_{W_m}$.

Because the response spectrum is real, obtain

$$S_x(\omega)=\left|H(\omega)\right|^2[S_{W_1}(\omega)+S_{W_2}(\omega)+...S_{W_m}(\omega)+2\,\mathrm{Re}(S_{W_1W_2}(\omega)+S_{W_1W_3}(\omega)...S_{W_{m-1}W_m}(\omega))]=$$
$$=\left|H(\omega)\right|^2[\sum_{j=1}^{m}S_{W_j}(\omega)+2\,\mathrm{Re}(\sum_{\substack{i=1\\j=1\\i<j}}^{m}S_{W_iW_j}(\omega))], \qquad (23)$$

where

$$\left|H(\omega)\right|^2=\frac{1}{m^2\left\{\left[p^2\left(1+\psi\frac{E\left\{x(t)\left(a_nx^n(t)+a_{n-1}x^{n-1}(t)+...+a_1x(t)+a_0\right)\right\}}{\sigma_x{}^2}\right)-\omega^2\right]^2+4\xi_e{}^2p_e{}^2\omega^2\right\}}. \qquad (24)$$

For completely uncorrelated processes, we have

$$S_{W_iW_j}(\omega)=0,\ i\neq j \qquad (25)$$

and the power spectral density of the response is

$$S_x(\omega)=\left|H(\omega)\right|^2[\sum_{j=1}^{m}S_{W_j}(\omega)\ . \qquad (26)$$

The spectral density of the response is given by:

$$S_x(\omega)=\frac{\sum_{j=1}^{m}S_{W_j}(\omega)}{m^2\left\{\left[p^2\left(1+\psi\frac{E\{x(t)\left(a_nx^n(t)+a_{n-1}x^{n-1}(t)+...+a_1x(t)+a_0\right)}{\sigma_x{}^2}\right)-\omega^2\right]^2+4\xi_e{}^2p_e{}^2\omega^2\right\}}. \qquad (27)$$

For $Q(x)=0$, obtain the linear case:

$$p_e=p,\xi_e=\xi,\alpha=0, \qquad (28)$$

$$S_x(\omega)=\frac{\sum_{j=1}^{m}S_{W_j}(\omega)}{m^2[\left(p^2-\omega^2\right)^2+4\xi^2p^2\omega^2]}. \qquad (29)$$

2. THE NUMERICAL RESULTS

For the spectral density of the excitations $S_{W_1}=S_{W_2}=2\ N^2\cdot s$, $m=1kg$, $k=36\frac{N}{m}$, $c=4\frac{Ns}{m}$, $\psi=7m^{-1}$, n=4, we will find the statistical parameters of function.

We obtain

$$p = \sqrt{\frac{k}{m}} = 6s^{-1},\ \frac{c}{m} = 2\xi p \Rightarrow \xi = 0{,}33\,. \qquad (30)$$

For the variance of the process *W*(*t*), we obtains

$$\sigma^2{}_x = 1{,}01 \cdot 10^{-2} m^2\,. \qquad (31)$$

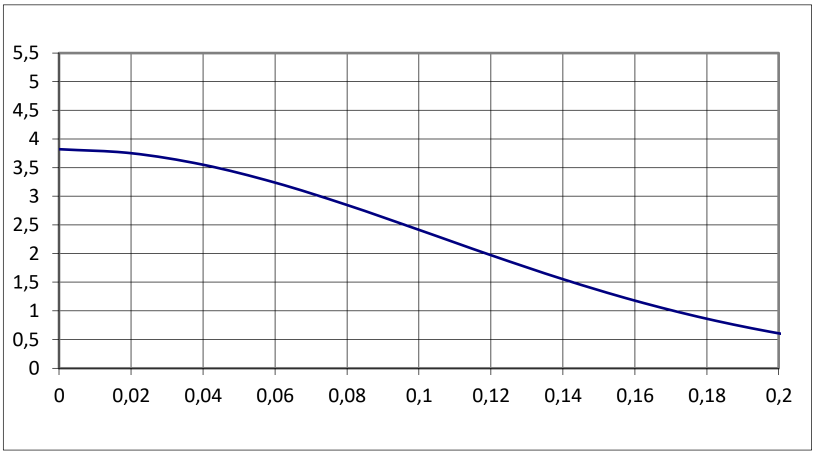
The undamped natural frequency for the linear system is

$$p_e{}^2 = p^2 \left(1 + \psi \frac{\sqrt{2\pi}\,\sigma_x (3\sigma_x{}^2 + 1)}{16} \right) = 6{,}329 s^{-1}. \qquad (32)$$

The probability density $P(x)\,[m^{-1}]$ are shown in fig. 1., for for $p = 6s^{-1}, \xi = 0{,}33$.

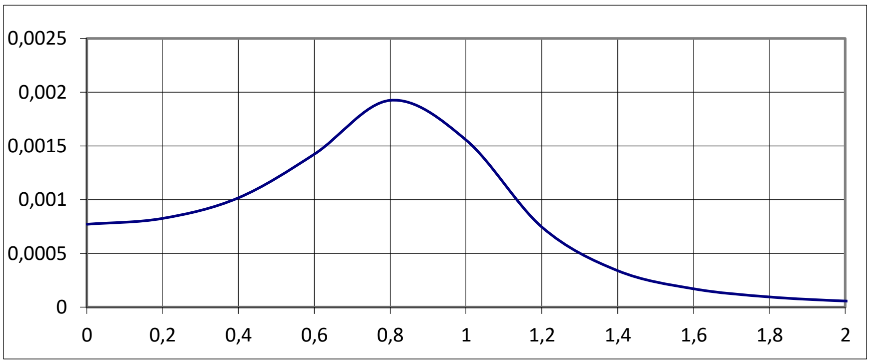
The power spectral density response are shown in fig. 2., for various parameter values.

The broadening of the first resonant peak is described very satisfactorily by the approximate solution. It should be noted that the presence of the 'extra' resonances becomes more evident for higher nonlinearities; for higher damping levels and a weak nonlinearity the higher resonances can almost disappear.



x [m]

Fig. 1. The probability density $P(x)\,[m^{-1}]$ for $p = 6s^{-1}, \xi = 0{,}33$.



$\omega / 2\pi$

Fig. 2. The power spectral density $S_x\,[m^2 \cdot s]$ for $m = 1kg,\ k = 36N/m,\ c = 4Ns/m,\ \psi = 7m^{-1}$.

3. CONCLUSIONS

This method has seen the broadest application because of their ability to accurately capture the response statistics over a wide range of response levels while maintaining relatively light computational burden. No general method is available at present to obtain the response probability density function and the power spectral density of a non-linear system under a given arbitrary Gaussian random input. It should be noted that the presence of the 'extra' resonances becomes more evident for higher nonlinearities; for higher damping levels and a weak nonlinearity the higher resonances can almost disappear. The results with respect to the power spectral density demonstrate another striking nonlinear response property; namely, the presence of a large amount of energy at low frequencies. Efficient equivalent linear systems with random coefficients for approximating the power spectral density can be deduced. The asymmetry of the nonlinearity is the cause for this phenomenon. The power spectral density of the response will not have a large spectral content at low frequencies and the skewness will be zero.

REFERENCES

[1] Roberts, J.B., Spanos, P.D., Stochastic averaging an approximate method of solving random vibration problems, International J. Non-Linear Mechanics., 21, 1986.
[2].Zhao, L. and Chen, Q., An equivalent non-linearization method for analyzing response of nonlinear systems to random excitations, Applied Mathematics and Mechanics, 1997.
[3].Stan, P., Analysis of single-degree of freedom non-linear structure under Gaussian white noise ground excitation, The Sciencific Book, University of Pitesti, 2009.
[4].P. Stan, M. Stan, The study spectral analysis to random vibrations for nonlinear oscillators,. International Congress of Automotive and Transport Engineering CAR 2017, Pitesti, Automotive Series, volume 27, 2017
[5].P.Stan, M. Stan, Nonlinear vibrations in automotive suspensions, International Conference Computational Mechanics and Virtual Engineering, Brasov, Romania, 2015
[6].M. Stan, P. Stan, Numerical solution of the proces of hydroplaning at wheelswith envelope, International Congress of Automotive and Transport Engineering, Mobility Engineering and Environment, CAR 2017, 017 IOP Conf. Ser. Mater. Sci. Eng. 252, 2017