High order beam elements for the stability and non-linear analysis of frame structures

Adam Dósa[1], Lucian Popa[1]

[1]*Department of civil engineering, "TRANSILVANIA" University, Braşov, 500152, Romania*

Summary

The Euler-Bernoulli beam elements presented in this paper are based on the improvement of the accuracy of the finite element solution by increasing the polynomial degree of the shape functions. This approach, called the p-version of the finite element method has become more and more attractive in the last two decades. It was shown in the literature, that the p-version for linear elliptic problems with smooth solutions converges exponentially in the energy norm. In the practical analyses, the p-version yields an accuracy which can hardly be obtained by the classical h-version, based on the refinement of the finite element mesh.

The elements studied in this paper show very high convergence in buckling and nonlinear analyses. The final result of the studies is a beam element with six degree polynomial shape functions, which yields engineering accuracy in the practical buckling and non-linear frame analyses with only one element per column. Although this type of element introduces three supplementary non-nodal degrees of freedom, the computational costs of the analyses are much lover than in the case of the classical beam elements based on cubic shape functions.

KEYWORDS: beam elements, p-version, buckling, non-linear analysis of frames.

1. INTRODUCTION

In this paper a family of Euler-Bernuolli beam elements are presented for stability and geometrically nonlinear structural analysis. In the nonlinear analysis the updated lagrangean formulation is used. The developement of the element equilibrium relations is based on the principle of virtual work. The element has six degrees of freedom at the nodes and a number of supplementary non-nodal displacements corresponding to higher order bending modes. In the current reference system, the diplacement and force vectors of the element are:

$$\mathbf{a} = (a_1, a_2, \cdots, a_{nge})^T = (u_1, v_1, \theta_1, u_2, v_2, \theta_2, a_7, \cdots a_{nge})^T,$$
$$\mathbf{f} = (f_1, f_2, \cdots, f_{nge})^T = (N_1, T_1, M_1, N_2, T_2, M_2, f_7, \cdots f_{nge})^T. \qquad (1)$$

Here *nge* is the total number of degrees of freedom of the element. The incremental and virtual displacements will be noted by $\delta\mathbf{a}$ and $\delta\mathbf{a}_v$. It is advantegeous to use a reduced set of displacement increments, removing the rigid body movement of the beam and using only strain-inducing displacements.

$$\delta\mathbf{a}_r = (\delta u_r, \delta\theta_{r1}, \delta\theta_{r2}, \delta a_7, \ldots \delta a_{nge})^T = (\delta u_r, \delta\mathbf{q}_r)^T. \qquad (2)$$

Where $\delta\mathbf{q}_r = (\delta\theta_{r1}, \delta\theta_{r2}, \delta a_7, \ldots \delta a_{nge})^T$ are the bending displacements, $\delta u_r = \delta u_{21} = \delta u_2 - \delta u_{1r}$ is the extension of the beam and

$$\delta\theta_{r1} = \delta\theta_1 - \alpha, \quad \delta\theta_{r2} = \delta\theta_2 - \alpha, \qquad (3)$$

are the reduced slopes $d\delta v/dx$. For small increments the rigid rotation of the beam, can be expressed as: $\alpha \cong \delta v_{21}/L = (\delta v_2 - \delta v_1)/L$ (figure 1).
Remark: Since the updated lagrangean formulation is used, on the current configuration only the bending displacements are nonzero.

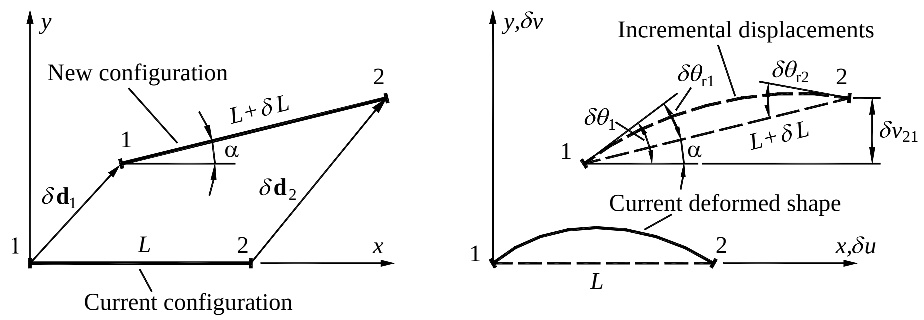

Figure 1. Incremental displacements of the beam

2. FORMULATION OF THE ELEMENTS

2.1 The shape functions

The transversal displacements of the beam are defined with the help of the shape functions. The first two functions are the conventional cubic shape functions (but with v_r=0 at the two ends). The next shape functions are hierarchical ones whitch give zero transversal displacements and slopes at the nodes of the beam.

$$v_r = \begin{Bmatrix} L/8(1-2\xi-4\xi^2+8\xi^3) \\ L/8(-1-2\xi+4\xi^2+8\xi^3) \\ (1-4\xi^2)^2 \\ L(1-4\xi^2)^2\xi \\ \vdots \\ L(1-4\xi^2)^2\xi^{nge-7} \end{Bmatrix}^T \begin{Bmatrix} \theta_{r1} \\ \theta_{r2} \\ a_7 \\ a_8 \\ \vdots \\ a_{nge} \end{Bmatrix} = \mathbf{n}\mathbf{q}_r . \tag{4}$$

Here $\xi = x/L - 1/2$. The polynomial order of these functions is up to $p = nge - 3$. The first derivative of (4) gives the slopes:

$$\theta = \frac{dv}{dx} = \frac{d\mathbf{n}}{dx}\mathbf{q}_r = \mathbf{s}\mathbf{q}_r . \tag{5}$$

Further derivation leads to the curvatures:

$$\chi = \frac{d\theta}{dx} = \frac{d\mathbf{s}}{dx}\mathbf{q}_r = \mathbf{b}\mathbf{q}_r . \tag{6}$$

The same relations will be used for the incremental and virtual quantities.

3. THE TANGENT STIFFNESS MATRIX

The equilibrium of the element in an incremental step can be expressed with the help of the principle of virtual work.

$$\delta V = \delta\mathbf{a}_v^T\mathbf{k}_t\delta\mathbf{a} - \delta\mathbf{a}_v^T\delta\mathbf{f} = 0. , \tag{7}$$

where $\mathbf{k}_t$ is the tangent stiffness matrix of the beam.

3.1 The stretching stiffness

In the incremental step, the bending and the stretching of the element are coupled because, the shape of the beam in the current configuration is curved. Since θ_r is small and $\delta\theta_r << \theta_r$ (see figure 2.a), the axial length increment due to the bending of the beam can be approximated by:

$$\delta L^q = \int_0^L d\delta L^q \cong \int_0^L \theta_r\delta\theta_r dx = \mathbf{q}_r^T\int_0^L \mathbf{s}\mathbf{s}^T dx\delta\mathbf{q}_r = \mathbf{q}_r^T\mathbf{k}^*\delta\mathbf{q}_r . \tag{8}$$

The virtual work done by the total axial stretching is:

$$\delta V^{s} = \delta \mathbf{a}_{rv}^{T} \begin{bmatrix} 1 \\ \mathbf{k}^{*}\mathbf{q}_{r} \end{bmatrix} \frac{EA}{L} \begin{bmatrix} 1 & \mathbf{q}_{r}^{T}\mathbf{k}^{*} \end{bmatrix} \delta \mathbf{a}_{r}. \tag{9}$$

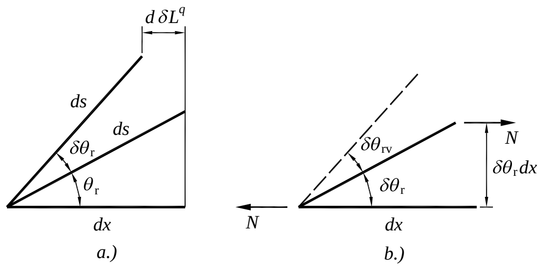

Figure 2. Higher order axial effects

3.2 The bending stiffness

The virtual work produced by the bending is composed by a term resulted from the moment-curvature part and a term given by the higher order effect of the axial force as the work done by the moment of *N* on the rotated infinitesimal element *dx* with the virtual rotation $\delta\theta_{v}$ (see figure 2.b).

$$\delta V^{b} = \int_{0}^{L} \delta\chi_{v} EI \delta\chi dx + \int_{0}^{L} \delta\theta_{v} N \delta\theta dx = \delta \mathbf{q}_{rv}^{T} (\mathbf{k}_{r} + N\mathbf{k}^{*}) \delta \mathbf{q}. \tag{10}$$

In relation (10)

$$\mathbf{k}_{r} = EI \int_{0}^{L} \mathbf{b}^{T}\mathbf{b} dx, \quad \mathbf{k}^{*} = \int_{0}^{L} \mathbf{s}^{T}\mathbf{s} dx. \tag{11}$$

Relation (11) can be evaluated symbolically. For example, for *nge*=9 (*p*=6) the following matrices result:

$$\mathbf{k}_{r(p6)} = \frac{EI}{L} \begin{bmatrix} 4 & 2 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1024/5 & 0 & 256/35 \\ 0 & 0 & 0 & 256/7 & 0 \\ 0 & 0 & 256/35 & 0 & 192/35 \end{bmatrix}, \tag{12}$$

$$\mathbf{k}^*_{(p6)} = \frac{L}{30}\begin{bmatrix} 4 & -1 & 16 & 24/7 & 4/7 \\ -1 & 4 & -16 & -24/7 & -4/7 \\ 16 & -16 & 1024/7 & 0 & 0 \\ 24/7 & -24/7 & 0 & 256/21 & 0 \\ 4/7 & -4/7 & 0 & 0 & 64/77 \end{bmatrix}. \tag{13}$$

3.2 Transferring the current nodal forces to the new configuration

The nodal forces in the current configuration are (see figure 3):

$$\mathbf{f}_n = (N_1, T_1, M_1, N_2, T_2, M_2)^T = (-N, T, M_1, N, -T, M_2)^T\,. \tag{14}$$

The shear force $T=(M_1+M_2)/L$ results from the equilibrium of the beam. In the new configuration the forces N and T change their direction and the shear force has a new value $T'=T+\Delta T$. From the new moment equilibrium of the current forces, retaining only the first order terms results:

$$\Delta T = -\frac{T\delta L}{L+\delta L} \cong -\frac{M_1+M_2}{L^2}\delta u_{21}. \tag{15}$$

The variation of the nodal forces expressed in the current reference system becomes:

$$\begin{aligned} \Delta X &= N\cos\alpha + T'\sin\alpha - N \cong T\sin\alpha \cong \frac{M_1+M_2}{L^2}\delta v_{21}, \\ \Delta Y &= N\sin\alpha - T'\cos\alpha - T \cong N\sin\alpha - \Delta T \cong N\frac{\delta v_{21}}{L} + \frac{M_1+M_2}{L^2}\delta u_{21}. \end{aligned} \tag{16}$$

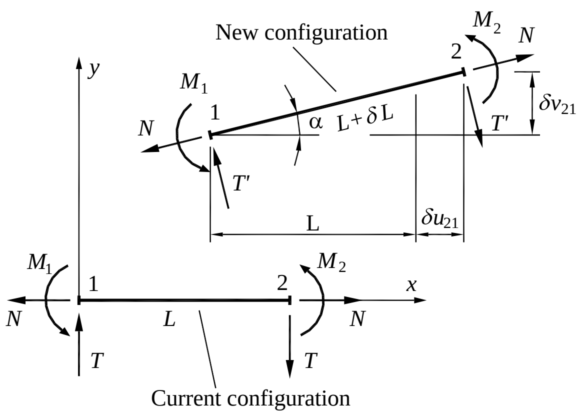

Figure 3. Transfer of the current nodal forces to the new configuration

The forces (16) and the virtual displacements of the nodes introduce a new virtual work term:

$$\delta V^t = \Delta X \delta u_{v21} + \Delta Y \delta v_{v21} = N \delta \mathbf{a}^T \mathbf{z}^T \mathbf{z} \delta \mathbf{a} + \frac{M_1 + M_2}{L^2}(\delta \mathbf{a}^T \mathbf{z}^T \mathbf{r} \delta \mathbf{a} + \delta \mathbf{a}^T \mathbf{r}^T \mathbf{z} \delta \mathbf{a}), \quad (17)$$

where $\mathbf{r}$ = (-1 0 0 1 0 0 0 ...0)^T and $\mathbf{z}$ = (0 -1 0 0 1 0 0 ... 0)^T, such that $\delta u_{21}=\mathrm{r}^T\delta\mathbf{a}$ and $\delta v_{21}=\mathbf{z}^T\delta\mathbf{a}$.

Using relations (7), (9), (10) and (17), the tangent stiffness matrix of the element can be written as:

$$\mathbf{k}_t = \mathbf{B}^T \begin{bmatrix} EA/L & \mathbf{0} \\ \mathbf{0} & k_r \end{bmatrix} \mathbf{B} + N\mathbf{B}^T \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & k^* \end{bmatrix} \mathbf{B} + \frac{N}{L} \mathbf{z}^T \mathbf{z} + \frac{M_1 + M_2}{L^2} (\mathbf{r}^T \mathbf{z} + \mathbf{z}^T \mathbf{r}). \quad (18)$$

In relation (18)

$$\mathbf{B} = \begin{bmatrix} \mathbf{r}^T + \mathbf{q}_r^T \mathbf{k}^* \mathbf{A} \\ \mathbf{A} \end{bmatrix}, \quad (19)$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & -1/L & 1 & 0 & 1/L & 0 & 0 & \cdots & 0 \\ 0 & -1/L & 0 & 0 & 1/L & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \quad (20)$$

such that $\delta\mathbf{q}_r = \mathbf{A}\delta\mathbf{a}$ extracts the reduced bending displacement set from the complete displacement vector of the element.

4. NUMERICAL EXPERIMENTS

4.1 Buckling of beams

Using the stiffness matrix (18) a convergence study was performed for the known cases of buckling of beams subjected to axial force. Both *h*- and *p*-refinements were used by dividing the beams and respectively increasing the polynomial degree of the elements. In the figure 4, the variation of the relative error $e_r = (N_{cr}^{approx} - N_{cr}^{exact}) / N_{cr}^{exact}$ of the buckling force for the worse case - the clamped-hinged beam is presented. The parameters of the corresponding refinements are presented in the table 1.

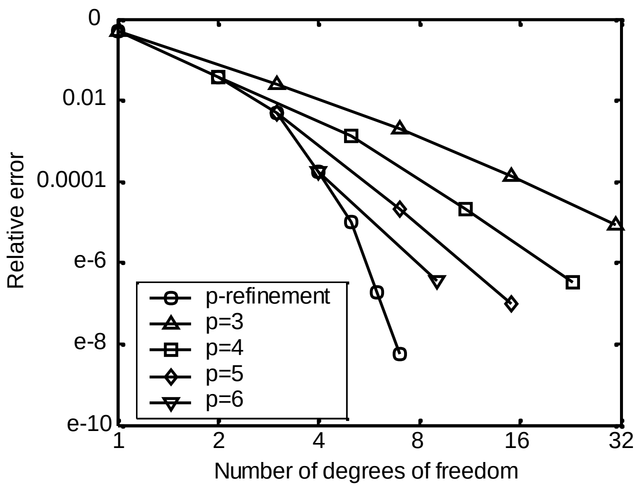

Figure 4. Convergence study for the buckling of a clamped-hinged beam

Table 1. Number of degrees of freedom for the beam buckling problem

Polynomial degree *p*	Number of divisions 1	2	4	8	16
3	1	3	7	15	31
4	2	5	11	23	
5	3	7	15		
6	4	9			
7	5				
8	6				
9	7				

It can be observed that in the case of the same number of degrees of freedom, the best results are obtained by the *p*-refinement.

4.1 Post-critical behavior of the Roorda frame

In the figure 5, the equilibrium paths are presented for six eccentric loadings of the Roorda's L-frame. This case is a classical example of asymmetric bifurcation, where for some small imperfections (introduced here by the eccentricity of the load) the post-critical limit load can be smaller than the corresponding critical load of the perfect system. For example for $e = -0.01L$, the limit load is $P_{max} = 0.899P_{cr}$ [1]. The result for this case obtained by using only two *p*6 elements, without dividing the beams was $P_{max}^{1-p6} = 0.899P_{cr}$.

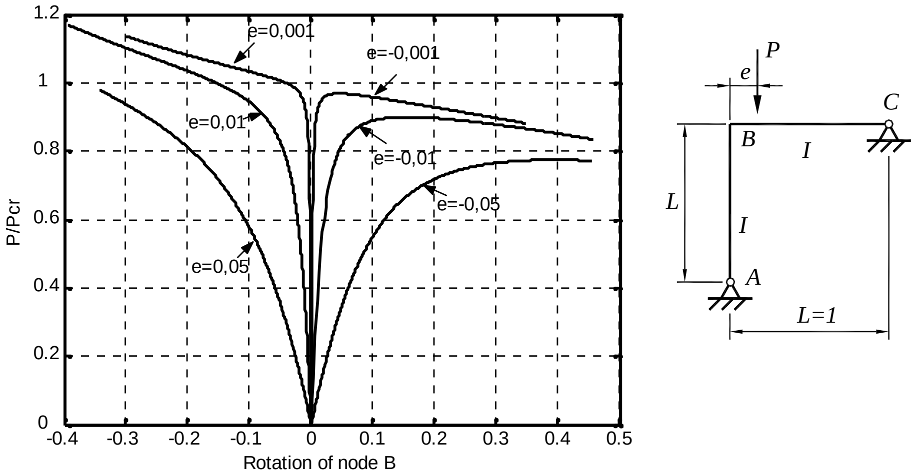

Figure 5. Equilibrium paths for the Roorda's L-frame

5. CONCLUSIONS

In this paper high order beam elements are presented for stability and geometrically nonlinear analysis of frames. The development of the elements uses non-nodal displacements and hierarchical shape functions with increasing polynomial degree for the bending of the elements. Numerical experiments show a very high accuracy of the results. In practical analyses, the element with polynomial degree six eliminates the necessity of dividing of beams and leads to engineering precision of results.

References

1. Bažant, Z.P., Cedolin, L. *Stability of structures: Elastic, Inelastic, Fracture and Damage Theories*, Dover Publications, 2003.
2. Bănuţ, V. *Calculul neliniar al structurilor*, Editura tehnică, Bucureşti, 1981.
3. Caracostea, A.D., ş.a. *Manual pentru calculul construcţiilor*, Editura tehnică, Bucureşti, 1977.
4. Przemieniecki, J.S. *Theory of matrix structural analysis*, Dover Publications inc., New York, 1985.
5. Crisfield, M.A. *Non-linear Finite Element Analysis of Solids and Structures*, John Wiley & Sons, 1991.
6. Szabó, B., Babuška, I. *Finite Element Analysis*, John Wiley and Sons Inc, 1991.
7. Zienkiewicz, O.C., Taylor, R.L. *The Finite Element Method, Volume 2: Solid Mechanics*, Fifth Edition, Butterworth-Heinemann, 2000.