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A NEW FINITE ELEMENT CONSIDERING SHEAR LAG

A. PROKIC[1] M. VOJNIC PURCAR[1] D. LUKIC[1]

Abstract: *A new model of describing the shear lag phenomenon in composite thin-walled beams with arbitrary open or closed cross sections is defined. This phenomenon is unable to calculate using the classical theory of thin-walled beams based on the assumption that shear strains in the middle surface can be neglected. Therefore, this paper is based on facts presented in the papers of Prokic. He proposed the new warping function valid for both, open and closed cross sections and it does not require assumption of neglecting shearing strains. The general approach to the solution of the problem is based on the finite element method. The principle of virtual displacements has been used to give a new linear stiffness matrix.*

Key words: *shear lag, finite element, thin-walled.*

1. Introduction

Thin-walled composite structures are widely used in many fields of aerospace, automotive, nautical and other industries. Over a past few decades they became broadly adopted in civil engineering due to many advantages of this material, like lightweight feature in relation of resistance, corrosion resistance, low thermal expansion, good mechanical characteristics, etc.

This significant increase in the use of thin-walled composite structures requests comprehensive analysis approach and many researchers work on this theme but, to the author's knowledge, only few of them dealt with the phenomenon known as shear lag. Shear lag effect may bring a non-uniform distribution of normal stresses in the beams, different from that predicted by the Bernoulli hypothesis. Ignoring this effect in the analysis of the mechanical behavior of thin-walled structures can lead to overestimated values of capacity, unacceptable from the standpoint of structural safety. This suggests that the effect of shear lag must be paid special attention.

The phenomenon of shear lag has been extensively studied in order to develop a reliable model for its analysis. The classical theory of thin-walled beams [1] is based on the assumption that the shear strains in the middle surface can be neglected. While this results offer a simple analytical solution, it is unable to reflect phenomenon such as shear lag.

Reissner [2] developed method based on the principle of minimum of potential energy to describe shear lag phenomenon. Moffat and Dowling [3] used finite element method to describe effective breadth concept, they first set up design rules for steel box girders, based on effective breath.

[1] Faculty of Civil Engineering Subotica, University of Novi Sad.

Papers dealt with investigation of shear lag in composite materials are much less represented. Some solutions to this issue are presented in the works of Takayanagi [4] and Lopez-Anido and GangPao's [5]. They examined the influence of shear lag on the I beams, and the thin-walled prismatic orthotropic composite beams. Recent paper was presented by Wu [6], he proposed solution of single-cell thin-walled composite-laminated box beams under bending loads with consideration of both shear lag and shear deformation. The lack of this solution is limited use. It is aplicable only on a symmetric composite single-cell box beams.

In this paper, the finite element describing the shear lag phenomenon is presented. It is defined on the basis of the warping function presented by Prokic [7,8]. This warping function is valid for open and close cross-sections. The assumption of neglecting the shear strain in the middle plane is not necessary, shear stresses can be directly determined from the relevant strains. The distribution of normal stresses caused by deplanation is not specified by warping function but the displacement parameters of nodal points. This allows analysis the influence of shear lag effect on girders.

2. Basic theory

A straight, thin-walled beam with an open or closed cross section is considered. The midline of cross-section is idealized by a number of straight lines connected by discrete points (nodal points of cross-section) i=1,2,...,n.

As usual, the two coordinate systems are used in the analysis of thin-walled beams. Descartes' coordinate system *xyz*, of the right orientation, where the *z* axis is parallel to the axis of the rod, and *x* and *y* axis lie in the cross section plane, and the curvilinear coordinate system *esz*, also of the right orientation, with unit vectors *n*, *t* and i_z, Fig.1.

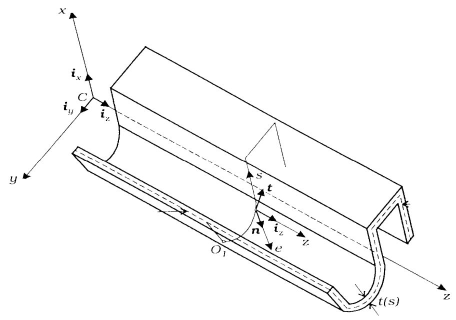

Fig. 1. *Thin-walled beam of arbitrary cross section*

The present theory is based on the following assumptions:

1. the cross-section is perfectly rigid in its own plane,
2. the longitudinal displacements caused by warping vary linearly between any two adjacent nodal points
3. the relative warping in relation to the midline is qualitatively defined with the solution of Saint-Venant's torque.

According to the first assumption the cross-sectional behavior can be described by only three displacement components, two translations *u* and *v* and an angle of twist φ of center of gravity (Fig. 2). From geometric considerations, normal and tangential displacements of an arbitrary point *S* with coordinates *x* and *y* on the contour, where the angle of twist is sufficiently small, are

$$\begin{aligned}\xi_* &= v\sin\alpha + u\cos\alpha + \varphi h_n \\ \eta_* &= v\cos\alpha - u\sin\alpha + \varphi h\end{aligned} \tag{1}$$

where α denotes the angle between the *x* and *n* axes, h_n represents the

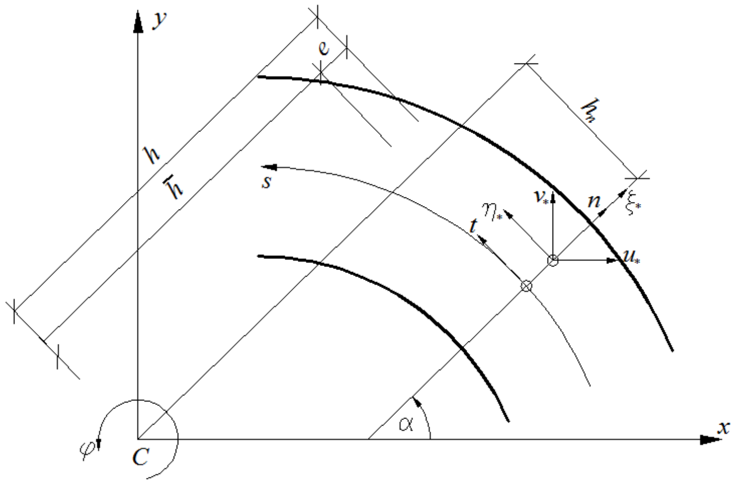

Fig.2. *Displacement component*

perpendicular distance from normal at point *S* to the point *C* given by

$$h_n = x\sin\alpha - y\cos\alpha \tag{2}$$

and *h* represents the perpendicular distance from tangent at point *S* to the point *C* given by

$$h = x\cos\alpha + y\sin\alpha \tag{3}$$

h_n and *h* are positive when normal *n* and tangent *t* respectively are rotating counterclockwise about the center of

gravity, when observed from positive *z* direction.

Displacement of cross-section at *z* direction can be described in the following form:

$$w_* = w + y\psi_x - x\psi_y + w_{warp} \tag{4}$$

The last term of (4) defines warping of the cross-section as suggested by Prokic [7,8].

$$w_{warp} = w_{warp}^s + w_{warp}^e \tag{5}$$

where

$$w_{warp}^s = \sum_i w_i(z)\,\Omega^i(x,y) \tag{6}$$

represents warping along the midline of cross-section. Unknown parameters w_i are displacements of arbitrary points on the midline. Those points are nodes of the section and their number determines the number of unknown parameters of displacements.

Function Ω^i depends on the mode of displacement change between the nodes of polygonal cross-section. If this change is linear, according to the second assumption, which is in conformity with the classical theory of thin-walled beams, then the function Ω^i has a simple geometrical meaning, as shown in Fig. 3. The function Ω^i exists only along parts between the point *i*, where it takes the value 1, and adjacent nodes, where it takes the value 0.

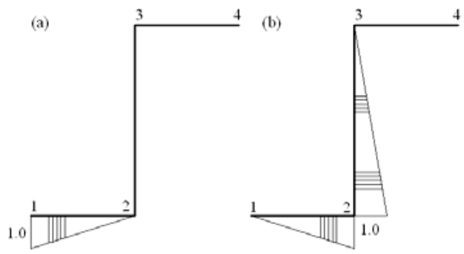

Fig. 3. *Warping function*

The second term on the right side of (5) determines the relative warping in relation to the midline of the cross-section, and, according to the third assumption, is equal to:

$$w_d^e = -\omega(x, y)\varphi'(z) \quad (7)$$

where

$$\omega(x, y) = h_n e \quad (8)$$

Now, for the total longitudinal displacement we obtain:

$$w_* = w + y\psi_x - x\psi_y + \sum_i \Omega^i w_i - \omega\varphi' \quad (9)$$

3. Finite element

A typical thin-walled element is shown in Fig.4. The element has 6+n degrees of freedom at each end node $u_i, v_i, w_i, -v_i', u_i', \varphi_i, w_{1i}, w_{2i}, \ldots, w_{ni}$

Equation (1) and (9) can be converted to matrix form

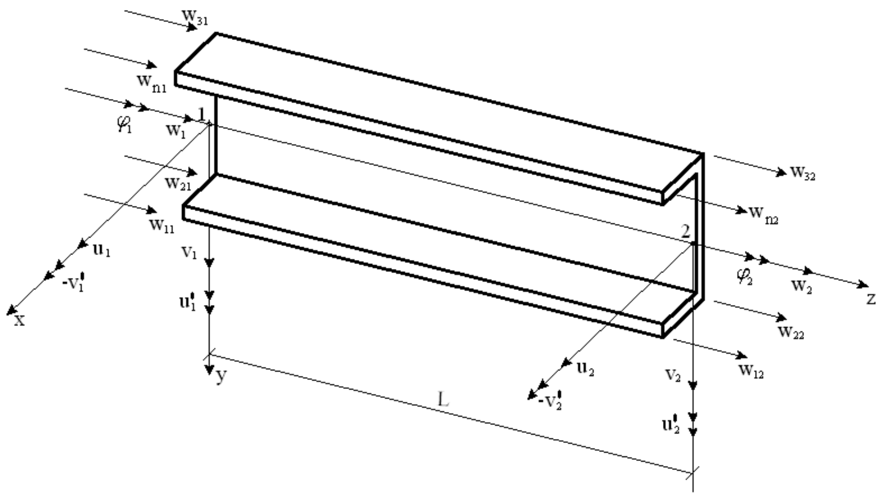

Fig.4. *Finite element*

$$\begin{bmatrix} \xi_* \\ \eta_* \\ w_* \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 & h_n & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ -\sin\alpha & 0 & \cos\alpha & 0 & h & 0 & 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & -x & 0 & -y & 0 & -\omega & 1 & \Omega^1 & \dots & \Omega^i & \dots & \Omega^n \end{bmatrix} \begin{bmatrix} u \\ u' \\ v \\ v' \\ \varphi \\ \varphi' \\ w \\ w_1 \\ \vdots \\ w_i \\ \vdots \\ w_n \end{bmatrix} \tag{10}$$

Let us denote the vector of generalized nodal displacement (Fig. 4) in the following way

$$q = \left[q_u, q_v, q_\varphi, q_w, q_{w_1}, q_{w_i}, \dots, q_{w_n} \right]^T \tag{11}$$

Where

$$\begin{aligned} q_u &= \left[u_1, u_1', u_2, u_2' \right]^T \\ q_v &= \left[v_1, -v_1', v_2, -v_2' \right]^T \\ q_\varphi &= \left[\varphi_1, \varphi_2 \right]^T \\ q_w &= \left[w_1, w_2 \right]^T \\ q_{w_i} &= \left[w_{i1}, w_{i2} \right]^T \qquad i = 1,2,\dots,n \end{aligned} \tag{12}$$

Hermitt polinomials are adopted as a interpolation functions for displacements u and v, and a linear displacements function is adopted for φ, w, $w_1, \dots, w_n$

$$\begin{aligned} N_u &= \left[1 - 3\xi^2 + 2\xi^3 \quad L\left(\xi - 2\xi^2 + \xi^3\right) \quad 3\xi^2 - 2\xi^3 \quad L\left(-\xi^2 + \xi^3\right) \right] \\ N_v &= \left[1 - 3\xi^2 + 2\xi^3 \quad L\left(-\xi + 2\xi^2 - \xi^3\right) \quad 3\xi^2 - 2\xi^3 \quad L\left(\xi^2 - \xi^3\right) \right] \\ N &= \left[1 - \xi \quad \xi \right] \end{aligned} \tag{13}$$

Then, we can write

$$\begin{aligned} u &= N_u q_u \\ v &= N_v q_v \\ \varphi &= N q_\varphi \\ w &= N q_w \\ w_i &= N q_{w_i} \quad i = 1,2,....,n \end{aligned} \qquad (14)$$

Substituting (14) into (10) displacement of an arbitrary point of cross-section could be obtained in terms of nodal parameters

$$\begin{bmatrix} \xi_* \\ \eta_* \\ w_* \end{bmatrix} = \begin{bmatrix} \cos\alpha N_u & \sin\alpha N_v & h_n N & 0 & 0 & ... & 0 & \dots & 0 \\ -\sin\alpha N_u & \cos\alpha N_v & hN & 0 & 0 & ... & 0 & \dots & 0 \\ -xN'_u & -yN'_v & -\omega N' & N & \Omega^1 N & ... & \Omega^i N & \dots & \Omega^n N \end{bmatrix} \begin{bmatrix} q_u \\ q_v \\ q_\varphi \\ q_w \\ q_{w_1} \\ \vdots \\ q_{w_i} \\ \vdots \\ q_{w_n} \end{bmatrix} = Aq \qquad (15)$$

where

$$\begin{aligned} N'_u &= \frac{1}{L}\left[-6\xi + 6\xi^2 \quad L\left(1 - 4\xi + 3\xi^2\right) \quad 6\xi - 6\xi^2 \quad L\left(-2\xi + 3\xi^2\right)\right] \\ N'_v &= \frac{1}{L}\left[-6\xi + 6\xi^2 \quad L\left(-1 + 4\xi - 3\xi^2\right) \quad 6\xi - 6\xi^2 \quad L\left(2\xi - 3\xi^2\right)\right] \\ N' &= \frac{1}{L}\left[-1 \quad 1\right] \end{aligned} \qquad (16)$$

4. Stiffness matrix

Considering assumptions, strain components different from zero are:

$$\varepsilon = \left[\varepsilon_z \quad \gamma_{zs} \quad \gamma_{zn}\right]^T \qquad (17)$$

where

$$\begin{aligned} \varepsilon_z &= \frac{\partial w_*}{\partial z} \\ \gamma_{zs} &= \frac{\partial \eta_*}{\partial z} + \frac{\partial w_*}{\partial s} \\ \gamma_{zn} &= \frac{\partial \xi_*}{\partial z} + \frac{\partial w_*}{\partial e} \end{aligned} \qquad (18)$$

Substituting (16) into (18) we obtain

$$\varepsilon = B \cdot q \qquad (19)$$

Where

$$B=\begin{bmatrix} 0 & 0 & \frac{\partial}{\partial z} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial s} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial e} \end{bmatrix} A=\begin{bmatrix} -xN''_u & -yN''_v & -\omega N'' & N' & \Omega^1 N' & \ldots & \Omega^i N' & \ldots & \Omega^n N' \\ 0 & 0 & \left(\bar{h}+2e\right)N' & 0 & \Omega^1_{,s} N & \ldots & \Omega^i_{,s} N & \ldots & \Omega^n_{,s} N \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \ldots & 0 \end{bmatrix} \tag{20}$$

where

$$\begin{aligned} N''_u &= \frac{1}{L^2}\left[-6+12\xi \quad L\left(-4+6\xi\right) \quad 6-12\xi \quad L\left(-2+6\xi\right)\right] \\ N''_v &= \frac{1}{L^2}\left[-6+12\xi \quad L\left(4-6\xi\right) \quad 6-12\xi \quad L\left(2-6\xi\right)\right] \\ N'' &= \left[0 \quad 0\right] \end{aligned} \tag{21}$$

We denote matrix of reduced stiffnesses with D

$$D=\begin{bmatrix} \overline{\overline{Q_{11}}} & \overline{\overline{Q_{16}}} & \\ \overline{\overline{Q_{16}}} & \overline{\overline{Q_{66}}} & \\ & & \overline{\overline{Q_{55}}} \end{bmatrix} \tag{22}$$

Linear stiffness matrix may be represented in the following form

$$K=\left[\begin{array}{c|c|c|c|c|c|c} I_{xx}K_1 & I_{xy}K_2 & -I_{xe}K_4 & -S_xK_4 & \cdots & \begin{array}{c}-I_{x\Omega}K_4 - \\ -I^1_{x\Omega}K_{10}\end{array} & \cdots \\ \hline & I_{yy}K_3 & -I_{ye}K_5 & -S_yK_5 & \cdots & \begin{array}{c}-I_{y\Omega}K_5 - \\ -I^1_{y\Omega}K_{11}\end{array} & \cdots \\ \hline & & I^1_{ee}K_6 & S_eK_6 & \cdots & \begin{array}{c}I_{\Omega e}K_6 + \\ +I^1_{\Omega e}K_7\end{array} & \cdots \\ \hline & symm & & FK_6 & \cdots & \begin{array}{c}S_{\Omega}K_6 + \\ +S^1_{\Omega}K_7\end{array} & \cdots \\ \hline & & & & \ddots & \vdots & \\ \hline & & & & & \begin{array}{c}I_{\Omega\Omega}K_6 + I^1_{\Omega\Omega}K_7 + \\ +I^1_{\Omega\Omega}K_8 + I^3_{\Omega\Omega}K_9\end{array} & \\ \hline & & & & & \vdots & \end{array}\right] \tag{23}$$

5. Numerical example

To test the accuracy of the proposed method a numerical example was analysed. A simply supported girder of cross-section shown in Fig.5 was subjected to a moment of torsion. Displacements of the centroid and shear center are shown in Table 1. Results show there is no need for leading in the shear centre because the displacements are close to zero.

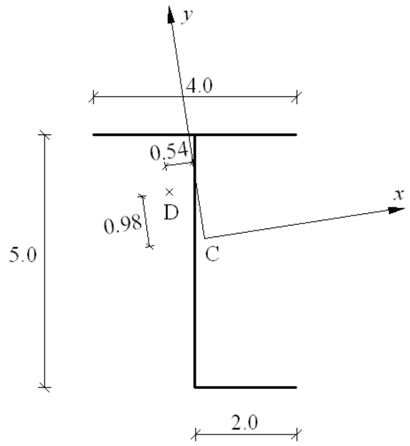

Fig.4. *Cros- section*

	Displacement in x direction	Displacement in y direction
centroid	2.778	1.523
Shear center D	-0.115 ≈ 0	-0.071 ≈ 0

Table 1
Displacements of the centroid and shear center

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