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COMMENTS ON THE SOLITON-LIKE INTERACTIONS IN NONDISPERSIVE MEDIA

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Abstract: *Seymour and Varley [1] analyse certain media whose responses are governed by the nonlinear nondispersive wave equation. Any two pulses traveling in opposite directions interact nonlinearly for a finite time when they collide but then part unaffected by the interaction. More clearly, when any two pulses are traveling in opposite directions meet and interact they emerge from the interaction region unchanged by the interaction. This interaction is similar to those that occur when two solitons collide. The main difference is that solitons are represented by waves of permanent form whose profiles are specific.*
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1. INTRODUCTION

In this work we present some comments on the properties of a coupled system of wave equations discussed by Seymour and Varley in [1]

$$y_{tt} = A^2(y_x)y_{xx} \ , \tag{1}$$

$$A_{,y_x} = A^{3/2}(\mu + \nu A) , \tag{2}$$

that govern the motion of an heterogeneous string, where $y(x,t)$ is the physical displacement, $A(y_x)$ a positive function representing the local speed of propagation, and μ,ν the material constants. If $y_x = y_x(x)$ and $A(y_x(x)) = A(x)$, the above system of equations becomes

$$y_{tt} = A^2(x)y_{xx} , \tag{3}$$

$$A_x = A^{3/2}(\mu + \nu A)y_{xx} \ . \tag{4}$$

We show that the waves described by Seymour and Varley are dispersive and dissipative. Single bounded solution of (2) may be written under the form

$$A(y_x) = \frac{\lambda[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]}{1 + \rho[e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]} , \tag{5}$$

with λ,ρ constants depending on μ,ν, and e_i, $i = 1,2,3$ the solutions of the equation $4y^3 - g_2 y - g_3 = 0$ with constants g_2, g_3 depending on μ,ν. The solution (5) shows cnoidal dependence on y_x, being characterized by the dependence of the amplitude on the argument of sn. For a certain values of μ,ν, for that $m = 1$, $m = \frac{e_2 - e_3}{e_1 - e_3}$, the solution (5) becomes

$$A(y_x) = \frac{\lambda[e_1 - (e_1 - e_3)\mathrm{sech}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]}{1 + \rho[e_1 - (e_1 - e_3)\mathrm{sech}^2(\sqrt{e_1 - e_3}\, y_x + \delta')]} . \tag{6}$$

The solutions of Seymour and Varley equations are also characterized by the dependence of the amplitude on the argument sech . The interaction (collision) of two solutions such (5) or (6) has solitonic properties: may propagate without change of form, being regarded as a local confinement of the energy of the wave field; at the collision each may come away with the same character as it had before the collision [2 -4]. Two solutions of (2) traveling in opposite directions interact nonlinearly and the collision is influenced by the properties of A .

2. LINEAR VIBRATING STRING

Consider 1D string motion equation

$$u_{tt} - c^2 u_{xx} = 0\,, \tag{7}$$

with c a real positive number. Let x range from $-\infty$ to ∞ . For the transverse vibrations of a string $c^2 = T/\lambda$ where T is the constant tension and λ the mass per unit length at the position x . For the compressional vibrations of an isotropic elastic solid in which the density and elastic constants are functions of x only (laminated medium) $c^2 = (\lambda + 2\mu)/\rho$. For the transverse vibrations of such laminated solid $c^2 = \mu/\rho$. The characteristics are given by $\mathrm{d}x/\mathrm{d}t = \pm c$, that are straight lines inclined to the axis at $c = \tan\varphi$. The D'Alembert solution of (7) is

$$u(x,t) = f(x - ct) + g(x + ct)\ , \tag{8}$$

where the functions $f, g : R \to R$ are determined from the initial conditions attached to (7)

$$u(x,0) = \Phi(x),\ u_{,t}(x,0) = \Psi(x)\ . \tag{9}$$

The solution (8) describes two waves $f(x - ct)$ and respectively $g(x + ct)$. The observer sees, at any moment of time the unchanged profile $f(x)$ at the initial time $t_0 = t'_0 = 0$. This is way the function $f(x - ct)$ represents a right travelling wave or a forward-going wave with the velocity c. For a similar reason, $g(x + ct)$ represents a left travelling wave or a backward-going wave with the velocity c. As a consequence, both waves are not interacting between them and do not change their shape during the propagation. These waves can be superposed by a simple sum, because of the linearity of (7).

These waves can be called *solitary waves*, for the reason they are not changing their shapes during propagation process, and do not interact one with the other.

If (7) is not linear we have the case considered in [1] for which the superposition principle is not valid. If y_1, y_2 are two solutions of (1), the sum between them is not a solution of (1). The function $A(y_x(x))$ is a positive function that represents the local speed of the propagation.

3. NONLINEAR VIBRATING STRING

Consider the equation (1) that can governs the motion of an heterogeneous string, where y is the physical displacement, and $A(y_x)$ is a positive function representing the local speed of propagation and verifies

$$A_{,x} = A^{3/2}(\mu + \nu A) y_{xx}\,, \tag{10}$$

we obtain a coupled partial nonlinear differential equations for $y(x,t)$ and $A(x)$

$$y_{tt} = A^2(x) y_{xx}\ , \tag{11}$$

$$A_x = A^{3/2}(\mu + \nu A) y_{xx}\ . \tag{12}$$

The characteristics are given by $\mathrm{d}x/\mathrm{d}t = \pm A$, which equations define two congruences of curves in the (x,t) plane. First step is to straighten the characteristics of (11) for geometrical representation in a space-time plane. To do this we define the transformation [5, 6]

$$x \to u(x) = \int_0^x \frac{\mathrm{d}z}{A(z)}\ . \tag{13}$$

From (11) we obtain

$$y_{tt} - y_{uu} + \frac{c_u}{c} y_u = 0\,,\quad c(u(x)) = A(x)\,. \tag{14}$$

The function $c(u)$ is the transformed local speed. Strictly speaking we should not use the same symbols y in (12), because the function $y(x,t)$ of (11-12) is not the same function as the $y(u,t)$ of (14), but this is not likely to cause confusion if we remember that y may be regarded as a physical quantity in terms of (x,t) or (u,t).
The linearized form of (14) is

$$y_{tt} - y_{uu} + y_u = 0\,. \tag{15}$$

Introducing the harmonic wave $y(u,t) = \tilde{A}\exp\mathrm{i}(ku - \omega t)$ into (15), we obtain the dispersion relation

$$\omega^2 = \mathrm{i}k + k^2\,. \tag{16}$$

The phase velocity of the harmonic waves $y(u,t) = \tilde{A}\exp(tkb)\exp\mathrm{i}k(u - \frac{t}{2kb})$ is $c_p = \frac{1}{2kb} = \frac{-1}{\sqrt{2k(-k+\sqrt{1+k^2})}}$ and depends on k. In conclusion, the equation (11) is *dispersive and dissipative.* In a space-time plane in which u and t are Cartesian coordinates, the characteristics are $\mathrm{d}x/\mathrm{d}t = \pm 1$, and are straight lines inclined to the axis at 45^0.

4. BOUNDED SOLUTIONS

Consider the equation (1) written under the form

$$A'^2 = A^3(\mu + \nu A)^2\,, \tag{17}$$

where $A' = A_{,e}$ and $e = y_x$. Differentiating (17), we have

$$A'' = a_2 A^2 + a_3 A^3 + a_4 A^4\,, \tag{18}$$

where

$$a_2 = 1.5\mu^2\,,\; a_3 = 4\mu\nu\,,\; a_4 = 2.5\nu^2\,, \tag{19}$$

and $a_i > 0$, $i = 2,3,4$. We assume the solution of (18) in the form [7]

$$A = \frac{\lambda P(e)}{1 + \rho P(e)}\,, \tag{20}$$

where $\lambda \neq 0$ and $\rho \neq 0$ are arbitrary real constants, and $P(e)$ is the Weierstrass elliptic function satisfying the differential equation

$$P'^2 = 4P^3 - g_2 P - g_3 \tag{21}$$

with two invariants g_2 and g_3 which are assumed to be real and satisfy

$$g_2^3 - 27 g_3^2 > 0\,. \tag{22}$$

So, the exact periodic solutions can be written as

$$A(y_x) = \frac{\lambda P(y_x + \delta; g_2, g_3)}{1 + \rho P(y_x + \delta; g_2, g_3)}\,, \tag{23}$$

where δ is an integration constant with known quantities g_2, g_3, λ and ρ. The exact bounded periodic solution can be obtained by replacing the Weierstrass elliptic function by the Jacobean elliptic sine function using the formula

$$P(y_x + \delta; g_2, g_3) = e_3 + (e_2 - e_3)\mathrm{sn}^2(\sqrt{e_1 - e_3}\, y_x + \delta')\,, \tag{24}$$

where δ' is an arbitrary real constant, and e_i , $i = 1,2,3$ are real roots of the equation $4y^3 - g_1 y - g_2 = 0$ with $e_1 > e_2 > e_3$. From $\mathrm{cn}^2 + \mathrm{sn}^2 = 1$, we can express (24) in term of the cnoidal function cn.
Thus, it results that the exact bounded periodic solution of (18) is (5). The solitonic form of (5) is given by (6).
Let again consider the transformation (13). Equation (1) can be written in the form

$$\frac{d}{dt} v(t) = Lv(t) , \tag{25}$$

where

$$v = \begin{pmatrix} y_u \\ y_t \end{pmatrix}, \quad L = \begin{pmatrix} 0 & \partial_u \\ \partial_u - 2\gamma & 0 \end{pmatrix}, \quad 2\gamma(u) = \frac{c_u}{c} \ . \tag{26}$$

We calculate $v(t)$ by a sequence of linear transformations that reduce $v(t)$ to a perturbation of the pulse. For this we define the energy of the string

$$E = \frac{1}{2} \int_{-\infty}^{\infty} \frac{y_u^2 + y_t^2}{c(u)} \mathrm{d}u \ , \tag{27}$$

and take a point v in the phase-space (y_u, y_t)

$$v = \begin{pmatrix} q \\ p \end{pmatrix}, \quad q(u) = y_u(u,0) , \quad p(u) = y_t(u,0) \tag{28}$$

The inner product is derived from the energy quadratic form (27)

$$< v_1, v_2 >= (C^{-1}v_1, C^{-1}v_2) , \quad C = \begin{pmatrix} \sqrt{c(x)} & 0 \\ 0 & \sqrt{c(x)} \end{pmatrix}, \tag{29}$$

$$(v_1, v_2) = \frac{1}{2} \int_{-\infty}^{\infty} [q_1(u) q_2(u) + p_1(u) p_2(u)] \mathrm{d}u \ . \tag{30}$$

The operator L is screw-symmetric with respect to the inner product (29) and so there exists a one-parameter group $V(t)$ of orthogonal transformation determined by

$$V_{,t}(t) = LV(t) , \quad V(0) = 1 , \tag{31}$$

so that $v(t) = V(t)v$ is a solution of (31). The method is based on the decomposition of the phase-space (y_u, y_t) into a pair of complementary subspaces. This induces a decomposition of each initial datum into a forward propagating part and a backward-propagating part.
In the homogeneous case $(\gamma = 0)$, (1) is reduced to $y_{tt} - y_{uu} = 0$ and the solution are expressed as a sum of two waves $f(x-t)$ and $f(x+t)$ that propagate independently. In the heterogeneous case the both pulses are coupled by $\gamma \neq 0$ considered as a perturbation.
So, we take

$$V(t) = CR^{-1}\tilde{V}(t)RC^{-1} , \ \tilde{V}(t) = \exp(tL) , \tag{32}$$

$$L = CR^{-1}\tilde{L}RC^{-1} , \ \tilde{L} = \begin{pmatrix} -\partial u & -\gamma \\ \gamma & \partial u \end{pmatrix}, \tag{33}$$

with $R = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\sqrt{2} \\ \frac{1}{\sqrt{2}} & \sqrt{2} \end{pmatrix}$. It results that $\tilde{V}(t) = RC^{-1}V(t)CR^{-1}$, $\tilde{L} = RC^{-1}LCR^{-1}$. We see that $\tilde{L}$ can be written as a sum of two operators to separate the contribution of the coupling term $\gamma \neq 0$

$$\tilde{L} = \tilde{L}_0 + \Gamma , \quad \tilde{L}_0 = \begin{pmatrix} -\partial u & 0 \\ 0 & \partial u \end{pmatrix}, \ \Gamma = \begin{pmatrix} 0 & -\gamma \\ \gamma & 0 \end{pmatrix}. \tag{34}$$

In the homogeneous case we have $\Gamma = 0$ and

$$\tilde V(t)=\tilde U(t),\quad \tilde U(t)=\begin{pmatrix} T(t) & 0\\ 0 & T'(t)\end{pmatrix}, \tag{35}$$

where $T(t)$ is right translation by t

$$[T(t)f](s)=f(s-t)\ , \tag{36}$$

and $T'(t)=T(-t)$ is left translation by t. The initial conditions (29) can be written under the form

$$y(u,0)=\varphi(u)\ ,\ y_t(u,0)=\psi(u)\ . \tag{37}$$

For $\Gamma=0$ the solution of $y_{tt}-y_{uu}=0$ is written as the D'Alembert formula

$$y(u,t)=\frac{1}{2}[\varphi(u+t)+\varphi(u-t)]+\frac{1}{2}\int_{u-t}^{u+t}\psi(z)\mathrm{d}z\ . \tag{38}$$

Our aim is to obtain a similar formula for the inhomogeneous case $\Gamma\neq 0$.
For this we use the well-known perturbation formula [8]

$$\tilde V(t)=\tilde U(t)+\int_0^t \tilde U(t-s)\Gamma\tilde V(s)\mathrm{d}s\ . \tag{39}$$

From this we can obtain an infinite series for $\tilde V(t)$ by an iteration scheme

$$\tilde V^{(n+1)}(t)=\tilde U(t)+\int_0^t \tilde U(t-s)\Gamma\tilde V^{(n)}(s)ds\ ,\qquad \tilde V^{(0)}(t)=\tilde U(t)\ . \tag{40}$$

We take account that $\tilde V(t)=\exp(tL)$ from (32) maps forward-going data into forward-going data and backward-going data. So, we write

$$\tilde V(t)=\begin{pmatrix} \tilde V_{FF}(t) & \tilde V_{FB}(t)\\ \tilde V_{BF}(t) & \tilde V_{BB}(t)\end{pmatrix}. \tag{41}$$

Here, $\tilde V_{FF}$ maps forward-going data into forward-going data, $\tilde V_{FB}$ maps forward-going data into backward-going data, $\tilde V_{BF}$ maps backward-going data into forward-going data and $\tilde V_{BB}$ maps backward-going data into backward-going data. From $(35)_2$ and (40) we obtain for $\tilde V_{FF}$

$$\tilde V_{FF}(t)=T(t)-\int_0^t\int_0^{t_1} T(t-t_1)\gamma T'(t_1-t_2)\gamma T(t_2)dt_1dt_2+..... \tag{42}$$

The first term in (42) is simply translating a forward-going datum into a forward direction. The integrant $T(t-t_1)\gamma T'(t_1-t_2)\gamma T(t_2)$ translate a forward-going datum in the forward direction from time zero to time t_2 when it is reflected. On reflection it is multiplied by the local reflection coefficient γ, then translated backwards from time t_2 to time t_1, when it is reflected again, multiplied by γ and translated forwards from time t_1 to time t. So, the second term in (42) represents the contribution to the forward-going disturbance from all possible double reflections. The following terms in (42) consider third reflections and so on. Knowing this, we have

$$V(t)=\begin{pmatrix} V_{11}(t) & V_{12}(t)\\ V_{21}(t) & V_{22}(t)\end{pmatrix}, \tag{43}$$

with

$$V_{11}(t)=0.5[V_{FF}(t)+V_{BB}(t)+V_{FB}(t)+V_{BF}(t)]\ ,\quad V_{12}(t)=0.5[V_{BB}(t)-V_{FF}(t)+V_{FB}(t)-V_{BF}(t)]\ ,$$

$$V_{21}(t)=0.5[V_{BB}(t)-V_{FF}(t)+V_{BF}(t)-V_{FB}(t)]\ ,\quad V_{22}(t)=0.5[V_{FF}(t)+V_{BB}(t)-V_{FB}(t)-V_{BF}(t)]\ . \tag{44}$$

Taking account of the initial data (37) we have

$$\begin{aligned} y_t(u,t)=0.5\sqrt{c(u)}[V_{BB}(t)-V_{FF}(t)+V_{BF}(t)-V_{FB}(t)]c^{-1/2}\varphi'(u)+\\ +0.5\sqrt{c(u)}[V_{BB}(t)+V_{FF}(t)-V_{BF}(t)-V_{FB}(t)]c^{-1/2}\psi(u), \end{aligned} \tag{45}$$

and

$$y(u,t) = \varphi(u) + 0.5\sqrt{c(u)}\int_0^t [V_{BB}(t) - V_{FF}(t) + V_{BF}(t) - V_{FB}(t)]c^{-1/2}(u)\varphi'(u)dt_1 +$$
$$+ 0.5\sqrt{c(u)}\int_0^t [V_{BB}(t) + V_{FF}(t) - V_{BF}(t) - V_{FB}(t)]\psi(u)\mathrm{d}t_1. \quad (46)$$

When $c'(u) = 0$, we have

$$y(u,t) = \varphi(u) + \frac{1}{2}\int_0^t [T'(t_1) - T(t_1)]\varphi'(u)dt_1 + \frac{1}{2}\int_0^t [T'(t_1) + T(t_1)]\psi(u)dt_1 . \quad (47)$$

After integration by parts and a change of variable (47) yields to D'Alembert formula (40).

5. CONCLUSION

As conclusion, from (46) we see that the solution of $y_{tt} = A^2(x)y_{xx}$ is expressed in term of the $\sqrt{c(u)} = \sqrt{A(x)}$, where $x \to u(x) = \int_0^x \frac{\mathrm{d}z}{A(z)}$, and $A(x) = A(y_x(x))$ verifying the solutions (5) and (6). These solutions are expressed in term of cnoidal (or soliton) solutions. Therefore, we can emphasis that the collisions between the Seymour and Varley waves have cnoidal or soliton characteristics. These waves distort as they propagate, and are of arbitrary shape and amplitude. Since such media transmit waves that *do not remember the interaction process* they are called *DRIP media*.

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REFERENCES

[1] Seymour, B.R., Varley, E., *Exact solutions describing soliton-like interactions in a nondispersive medium*, Soc. Ind. Appl. Math..J. Applied Mathematics 42, 804-821, 1982.

[2] Munteanu, L., Donescu, St., *Introduction to Soliton Theory*: *Applications to Mechanics*, Book Series Fundamental Theories of Physics, vol.143, Kluwer Academic Publishers, Dordrecht, Boston (Springer Netherlands) 2004.

[3] Ionescu, M.F., Chiroiu, V., Dumitriu, D., Munteanu, L., *A special class of DRIP media with hysteresis*, Revue Roumaine des Sciences Techniques, série de Mecaniqué Appliquée, 56(1), 3-10, 2011.

[4] Zabusky and Kruskal, *Interactions of solitons in a collisionless plasma and the recurrence of initial states*, Phys, Rev. Lett, vol 15, 1965.

[5] Lewis. J. T., *The heterogeneous string: coupled helices in Hilbert space*, Quarterly of Applied Mathematics, XXXVIII, nr. 4, 1980.

[6] Synge, J. L., *On the vibrations of a heterogeneous string*, Quarterly of Applied Mathematics, XXXIX, nr. 2, 1981.

[7] Krishnan E. V.,*On the Ito-Type Coupled Nonlinear Wave Equation*, J. of the Physical Society of Japan, 55, 11, 1986, 3753-3755.

[8] Kato T., *Perturbation theory for the linear operators*, Springer, New York, 1961.