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# THE MECHANICAL BEHAVIOR AND THE MATHEMATICAL MODELING IN THE CASE OF THE VIBRATIONS INDUCED TO THE SPINE

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***Abstract:*** *Back injuries, especially lower back pain, are amongst the most prevalent and costly non-lethal medical conditions affecting adults. An association between vehicle whole body vibration exposure and the development of lower back pain has been established through numerous epidemiological studies. In this paper we present a concept which could absorb the vibrations that appear at the driver or the passenger of a car or bus. To protect and prevent the lower back pains generated by the car vibrations and the rough road we construct a vibroisolator that can be inserted in the seats. Under those circumstances the following actions are required: the control, adjustment of the vibrations, their extinction, absorption or isolation – through vibro-isolation systems – environment protection, of real estate property and machines, the capture (absorption) of the vibrations, regularization and automated control of the sources by instruments of measure and control.*
***Keywords:*** *spine injuries, vibrations, mathematical modeling, transfer function, stability*

## 1. INTRODUCTION

Intervertebral discs provide flexibility of the spine and transmit and distribute large loads through the spine. To carry out these tasks the intervertebral discs have a particularly complex structure consisting of a gelatinous nucleus pulpous (NP) and the annulus fibrous (AF). However, many people show degenerative changes in the intervertebral discs due to aging or pathological process. These changes affect the composition and structure of the intervertebral discs, and their mechanical functions too. Back pain is often a clinical consequence of disc degeneration.

The intervertebral disc is a complex structure, and its behavior is governed by its biochemical as well as mechanical composition. Simulation of the disc function is therefore challenging and has led to the development of a number of different approaches to represent its behavior, i.e. the NP has often been modeled as a non-linear incompressible solid governed by a Mooney-Rivlin law or a fluid , while the AF was modeled as a homogeneous, isotropic, linear-elastic solid. The highly layered and oriented structure of the AF suggests that its material behavior may be significantly anisotropic. The anisotropic behavior of the AF can be taken into account through discrete representation of the collagen fibers embedded within a homogeneous or hyperelastic matrix (the ground substance).

Back injuries, especially lower back pain, are amongst the most prevalent and costly non-lethal medical conditions affecting adults [1]. An association between vehicle whole body vibration exposure and the development of lower back pain has been established through numerous epidemiological studies [2].

In this paper we present a concept which could absorb the vibrations that appear at the driver or the passenger of a car or bus. To protect and prevent the lower back pains generated by the car vibrations and the rough road we construct a vibroisolator that can be inserted in the seats.

Under those circumstances the following actions are required: the control, adjustment of the vibrations, their extinction, absorption or isolation – through vibro-isolation systems – environment protection, of real estate property and machines, the capture (absorption) of the vibrations, regularization and automated control of the sources by instruments of measure and control.

## 2. MATHEMATICAL VIBROISOLATOR

The absorptions of the vibrations or their control, required ecologically or admitted for reliability, can be achieved through constructing mechanical, hydro-pneumatic or hydro-electric systems. These vibroisolators are implemented in the mechanical system, in general it is considered a large mass $M$ (platforms) which vibrates and near by smaller masses ( $m$ ) are installed – dynamic absorbers which are disrupted by $M$ , or the masses $m$ are connected at the amortization, resistance, friction systems (fig. 1) [4].
Usually, the vibration source of mass $M$ operated by the disturbing force $F(t)$ and the amortized mass $m$ must form a system of forces which acts on the same axis (usually the vertical) because the gravitation force is in most of the cases included in the component of the $F(t)$ force.

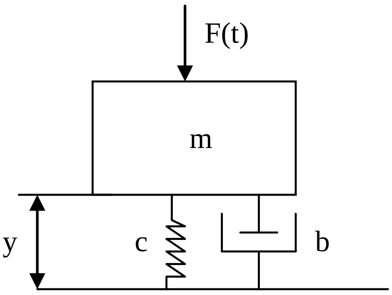


**Figure 1:** Mass-spring-damper mechanical model

The mass $M$ is the fundament, and the mass $m$ is the improvement-absorption source. The vibroisolator is located between $M$ and $m$, has a resistance coefficient $b$ and a vibro-isolation coefficient $c$ (vibro-isolation rigidity). The rigidity coefficient $c$ can be determined through equality conditions between the potential energy of the vibro-isolator and the equivalent elastic spring. This can be a non-linear function of coordinate $y$ (rate) calculated from the equilibrium position. The resistance coefficient $b$ is determined from the mechanical labor (energy) through the friction in the vibroisolator (amortization) and, in general, it can be a nonlinear function of $y$ and $\dot{y}$ .

$$m\ddot{y} = F(t) + Q(y, \dot{y}) \tag{1}$$

Here, $Q$ is the vibroisolator reaction. The purpose of the system is to minimize the components from $Q$ on the coaxial direction $y$ , with respect to the perturbation $F(t)$ .
We consider the linear vibroisolator:

$$Q(y, \dot{y}) = -cy - b\dot{y} \tag{2}$$

The movement equation is:

$$\ddot{y} + 2\gamma\dot{y} + \lambda^2 y = \frac{F(t)}{m} = \frac{M}{m} sin\omega t \tag{3}$$

with the solution

$$y = \frac{M}{m\sqrt{(\lambda^2 - \omega^2) + 4\gamma^2\omega^2}} sin(\omega t - \theta) \tag{4}$$

where:

$$tg(\theta) = \frac{2\gamma\omega}{\lambda^2 - \omega^2} \tag{5}$$

equivalent with:

$$\dot{y} = \frac{M\omega}{m\sqrt{(\lambda^2 - \omega^2) + 4\gamma^2\omega^2}} sin(\omega t - \theta) \tag{6}$$

From (4) and (5) we have the following relation in $Q$ :

$$Q = -K_{din} M \left[ sin(\omega t - \theta) + \frac{\omega b}{c} cos(\omega t - \theta) \right] \tag{7}$$

where $K_{din}$ is the dynamic coefficient of the perturbation, equal with the report of the sustained amplitude of the oscillations and the maximal value of movement $M/c$ for the static equilibrium of the force:

$$K_{din} = \frac{\lambda^2}{\sqrt{(\lambda^2 - \omega^2) + 4\gamma^2\omega^2}} \tag{8}$$

we represent (7) with the hint: $asin\alpha + bcos\alpha = \sqrt{a^2 + b^2} sin(\alpha + \varepsilon)$, with $\varepsilon = arctgb/a$. For $a = 1$ and $b = \frac{2\gamma\omega}{\lambda^2}$, we have:

$$Q = -K_{din} M \sqrt{1 + \frac{4\gamma^2\omega^2}{\lambda^4}} sin(\omega t - \theta + \varepsilon) \tag{9}$$

where $\varepsilon = arctg \frac{2\gamma\omega}{\lambda^2}$, therefore:

$$Q_{max} = K_{din} M \sqrt{1 + \frac{4\gamma^2\omega^2}{\lambda^4}} \tag{10}$$

The report $Q_{max}/M$ is called the transfer coefficient of the $K_c$ force and is the same as the dynamic coefficient:

$$K_c = \frac{Q_m ax}{M} = \sqrt{\frac{1 + 4\beta^2\gamma^2}{(1-\gamma^2)^2 + 4\gamma^2\beta^2}} \tag{11}$$

where $\gamma = \omega/\lambda$ is the frequency coefficient, and $\beta = \frac{\gamma}{\lambda}$ is the amortization (coupling) report. The transfer coefficient $K_c$ is characterizing the quality of the vibroisolation as follows: if the link between $m$ and $M$ is rigid, then $K_c = 1$; for $K_c < 1$ the vibroisolation is efficient; for $K_c > 1$ the vibroisolation becomes disturbing for foundation . Alongside $K_c$ (the forces transfer coefficient) it is also used the effective vibroisolation coefficient $K_{ef} = \frac{M}{Q_{max}} = \frac{1}{K_c}$

## 2. THE HYDROELECTRIC VIBROISOLATOR SYSTEM

The regulator systems, also called active control systems, are those vibroisolation systems in which the effective isolation against the vibrations is obtained by compensating the disturbing forces (disturbance based compensation) (fig. 2) Vibroisolator for the movement of the $m$ mass relative to the static position [4], [5].
In the case of the electrohydraulic vibroisolator we have the external perturbation is $F(t)$. The vibroisolator is composed from the rigidity $c$ and the damper $b$. The interaction (control) force $F_y$ is given by the reaction of the hydrocylinder 1 of which piston acts on the object $m$ through the rod 2 and the complementary elastic spring $c_y$. The movement of the piston through the pneumatic hydrocylinder is controlled with the help of the sensor signals 3 relative to the movement of the $m$ object and the piston. This signal is transmitted through the amplifier 4, with an electric supply 5. The convector (amplifier) processes the signal and transmits it to the regulator (the debit translator) 6 which adjusts the fluid movement from the convector (pump) 7 through the force cylinder. In the intermediate position both pipes are closed. Through the movement of the valve 6, the fluid under pressure in moving in the superior half of the cylinder and the piston goes down, if the valve goes down, the piston climbs, respectively.

The movement of the $m_g$ piston is $z$ and implies the variation of the elastic force $F_y = c_y(y - z)$ which acts on $m$ through the complementary spring $c_y$. Through an optimal choice of the vibroisolator parameters, the force $F_y$ acts opposed (reaction) to the force $F(t)$ and reduces the movement $y$. This force is called controlled (conducted) interaction force.

### 2.1. The non-holonomous link in the electro-hydrodynamic system

The generalized coordinates are chosen to be $y$ and $z$. This will determine a link that expresses the dependency of the piston speed $\dot{z}$ on the movement $y$. This link is defined by the variation of the fluid debit $Q_p$ which passes through a window of the drawer 6, [7].

$$Q_p = S_p \dot{z} \tag{12}$$

where $S_p$ is the effective area of the piston. We assume that the profile of the drawer's windows is chosen so that the fluid flow that enters in the cylinder is directly proportional with the drawer's opening.

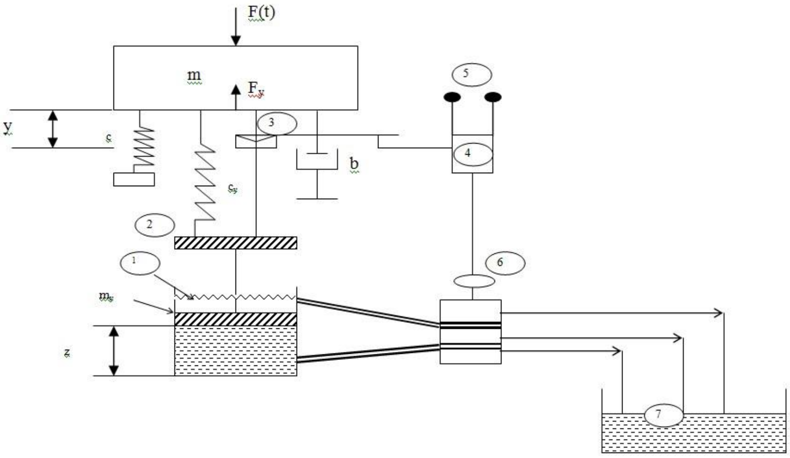


**Figure 2.** The electrohydraulic vibroisolator with the external perturbation is $F(t)$.

$$Q_p = K_{st} z_{st} \tag{13}$$

with $K_{st}$, the proportionality coefficient, depending on the drawer's parameters. Therefore, the electrical part of the system answers proportionally with the drawer's movement with respect to the movements $\varphi = y - z$ registered at 3:

$$z_{st} = K_\varphi (y - z) \tag{14}$$

$K_\varphi$ is a proportional coefficient depending on the parameters of the electrical machine. Hence, from (12), (13), (14) we have:

$$\dot{z} = K(y - z) \tag{15}$$

where $K = \dfrac{K_{st} K_p}{S_p}$. The equation (15) can not be integrated since we have a non-holonomous link.

### 2.2. The movement equations of the system

We have two variables in the system, $y, z$ but because of the link (15), through elimination we will have only one left degree of freedom left, $y$ or $z$ and therefore we'll obtain a differential equation. Further, we can write the movement equations, [3]:

$$\begin{aligned} &F(t) - cy - c_y(y - z) - b\dot{y} - m\ddot{y} = 0 \\ &c_y(y - z) - F_p - m_y \ddot{z} = 0 \end{aligned} \tag{16}$$

where $m_y$ is the piston mass and, $F_p$ the pressure force of the piston. We say that $\varphi = y - z$ and obtain the following system:

$$\dot{y} - \dot{\varphi} = K\varphi \qquad F(t) - cy - c_y\varphi - b\dot{y} - m\ddot{y} = 0 \qquad c_y\varphi - F_p - m_y K\dot{\varphi} = 0 \tag{17}$$

$$c_y \varphi = F(t) - cy - b\dot{y} - m\ddot{y} \tag{18}$$

$$c_y \dot{\varphi} = \dot{F}(t) - c\dot{y} - b\ddot{y} - m\dddot{y} \tag{19}$$

$$m\dddot{y} + (b + Km)\ddot{y} + (c + c_y + Kb)\dot{y} + cKy = KF(t) + \dot{F}(t) \tag{20}$$

Applying in (20) the Laplace transform with initial conditions we get:

$$[ms^3 + (b + Km)s^2 + (c + c_y + bK)s + cK]Y = (k + s)X \tag{21}$$

We define the transfer function $W$: $W = \dfrac{Y}{X}$

$$W = \frac{K + s}{ms^3 + (K + Km)s^2 + (c + c_y + bK)s + cK} = \frac{K + s}{P_3(s)} \tag{22}$$

If the function $F(t)$ is harmonic, meaning that $F = H\sin\omega t$ then $X = \dfrac{\omega}{s^2+\omega^2}$. In this situation we have:

$$W = \frac{\omega(K+s)}{(s^2+\omega^2)\left(ms^3+(K+Km)s^2+(c+c_y+bK)s+cK\right)} = \frac{\omega(K+s)}{P_5(s)} \tag{23}$$

In order to retrieve the original $y(t)$ we can discuss the roots of the polynomial $P_3(s)$ in the reduced form $P_3(s) = m(z^3+pz+q)$ replacing the unknown function with $s = z - \dfrac{a_1}{a_0}$.

The effective command (control) coefficient for the vibroisolation $K_{ef}$ will be defined as the report between the modulus $\left|y^o(i\omega)\right|$ which represents the complex amplitude without the vibroisolator interaction and $\left|y(i\omega)\right|$, the complex amplitude with the vibroisolator interaction.

$$K_{ef} = \left|\frac{y^o(i\omega)}{y(i\omega)}\right| \tag{24}$$

We choose in (23) $s = i\omega$ and we obtain the transfer function in the following form:

$$W(i\omega) = \frac{K+i\omega}{m(i\omega)^3+(K+Km)(i\omega)^2+(c+c_y+bK)(i\omega)+cK} \tag{25}$$

which can also be written as:

$$W(i\omega) = \frac{K+i\omega}{cK-(b+Km)(i\omega)^2+i\omega(c+c_y+bK-m\omega^2)} \quad \textbf{or} \quad W(i\omega) = \frac{(K+i\omega)(s_1-i\omega s_2)}{s_1^2+\omega^2 s_2^2} \tag{26}$$

where we chose: $s_1 = cK-(b+Km)\omega^2$, $s_2 = c+c_y+bK-m\omega^2$

$$W = U+iV, \qquad U = \frac{Ks_1+\omega^2 s_2}{s_1^2+\omega^2 s_2^2}, \qquad V = \frac{\omega(s_1-Ks_2)}{s_1^2+\omega^2 s_2^2} \tag{27}$$

$$\left|W(i\omega)\right| = \sqrt{U^2+V^2} = \frac{\sqrt{(Ks_1^2+\omega^2 s_2)^2+\omega^2(s_1^2-Ks_2)^2}}{s_1^2+\omega^2 s_2^2} \tag{28}$$

If the hydraulic system is without the vibroisolator action, then $K=0, c_y=0, z=0$. Hence, the movement equation of the damper becomes:

$$m\ddot{y}^o + b\dot{y}^o + cy^o = F(t) \tag{29}$$

getting the new transfer function:

$$W^o = \frac{1}{ms^2+bs+c} \tag{30}$$

So the sequential transfer function is:

$$W^o(i\omega) = \frac{1}{-m\omega^2+ib\omega+c}, W^o(i\omega) = U^o+iV^o, U^o = \frac{c-m\omega^2}{(c-m\omega^2)^2+b^2\omega^2}, V^o = \frac{-b\omega}{(c-m\omega^2)^2+b^2\omega^2} \tag{31}$$

$$\left|W^o(i\omega)\right| = \frac{1}{\sqrt{(c-m\omega^2)^2+b^2\omega^2}} \tag{32}$$

$$K_{ef} = \left|\frac{W^o(i\omega)}{W(i\omega)}\right| = s_1^2+\omega^2 s_2^2\sqrt{\frac{(c-m\omega^2)^2+b^2\omega^2}{(Ks_1+\omega^2 s_2^2)^2+\omega^2(s_1-Ks_2)^2}} \tag{33}$$

To optimize the vibroisolation we have the condition: $K_{din} < 1 \Rightarrow K_{ef} = \dfrac{1}{K_{din}} > 1$, hence the parameters $b, c, c_y, K$ are chosen so that we have: $K_{ef} > 1$. We can perform an analysis in the parameters space. For example: we settle $b, c, K$ and we can study the variation of $c_y$; we settle $c, K$ and we can study the variation of the parameters $b, c_y$; for $b = 0$ the damper is missing; for $c = 0$ the spring is missing.

## 3. THE STABILITY STUDY

In the movement equation (20) we consider that the free term $F(t)+K\dot{F}(t)$ is limited because for $F = Hsin\omega t$ the general solution is given by: $y(y) = y_g^o + y_p^n$, where $y_p^n$ is limited. We are interested in the case of the homogenous equation when the free term is not in resonance with the frequency.

### 3.1. The stability with the Routh – Hurwitz criterion

$$a_0 r^3 + a_1 r^2 + a_2 r + a_3 = 0, \qquad a_0 = m, a_1 = b + Km, a_2 = c + c_y + bK, a_3 = cK \tag{34}$$

In order for the system to be stabile it is necessary to have:

$$a_i > 0, \qquad a_1 a_2 > a_0 a_3 \Leftrightarrow (b+Km)(c+c_y+bK) > mcK$$

If we consider $b = 0$ then the damper is missing, we have $c + c_y > c$ therefore in order to study the stability the spring must exist.

• If $c_y = 0$ we have full stability $bc + K^2 mb + b^2 K > 0$

• The general case is satisfied because the real part of the roots of the characteristic equation is negative $\Re(r_k) < 0$

• The resonance case: If the algebra equation would admit $\pm i\omega$ as roots, then we obtain an impossible condition: $-bc_y - bK^2 = \omega^2 b$. Therefore, the system can not have resonance.

### 3.2. The stability with the Nyquist criterion

This criterion is used for the analysis or the structure, functional and spectral analysis of the system.

The vibroisolator system is a linear system with a link of input-output type, where the report $\frac{y}{x}$ implies the analysis of $W = \frac{Y}{X}$. In this system the input quantity $x(t)$ is added to or subtracted from $x_y(t)$ (the piston – spring interaction $F_y$). The signal is transmitted to the output $y(t)$ through the transfer function $W_i$:

$$W_i = \frac{Y}{X \pm X_y} \tag{35}$$

The quantity $y(t)$, once found, is used to retrieve the value of $F_y$ and so, the value of $x_y(t)$ that becomes $X_y$ is determined by constructing the transfer function for the inverse link:

$$W_l = \frac{X_y}{Y} \tag{36}$$

The inverse link is called positive for $x + x_y$ and negative for $x - x_y$. The transfer function for the entire system is obtained after replacing $Y$ from (36) and $X$ from (35):

$$W = \frac{X_y / W_l}{(Y \mp X_y W_i)/W_i} = \frac{W_i}{1 \mp W_l W_i} \tag{38}$$

The signs are chosen in the following manner: - for a system with a positive link and + for a system with a negative link. Hence, the vibroisolator system looks like:

$$x = F(t), \qquad x_y = c_y(y-z) = c_y \varphi \tag{39}$$

The movement equation becomes:

$$F(t) - cy - c_y\varphi - b\dot{y} - m\ddot{y} = 0 \tag{40}$$

To this equation we apply the Laplace transform and we obtain:

$$(ms^2 + bs + c)Y = X - X_y \Rightarrow W_i = \frac{Y}{X - X_y} = \frac{1}{ms^2 + bs + c} \tag{41}$$

$$sY = K\frac{X_y}{cy} + s\frac{X_y}{c_y} \quad W_i = \frac{X_y}{Y} = \frac{c_y s}{s+k}$$

In conclusion, the transfer function has the following form:

$$W = \frac{s+K}{(s+K)(ms^2+bs+c)+c_y s} \tag{42}$$

For systems without the vibroisolator action $W^o = W_i$ the following coefficient is obtained:

$$K_{ef} = \left|\frac{W_i}{W}\right| = \left|1+W^o(i\omega)W_i(i\omega)\right| = \left|1+\frac{c_y i\omega}{(i\omega+s)(-m\omega^2+bi\omega+c)}\right|$$

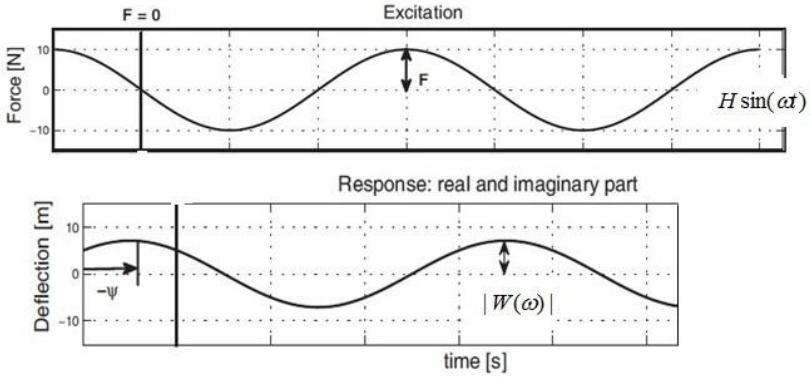


**Figure 3.** Excitation and response as a function of time.

## 4. CONCLUSION

In fig. 3 the excitation is a sinusoidal function with frequency ω and amplitude F. The bottom graph shows the complete response, which is a sinusoidal with the same frequency ω. The response characteristic has shifted compared to the excitation characteristic over the phase angle ψ. Here can be observed that the real part (Re, second panel) is in-phase with the excitation force, and that the imaginary part (Im, third panel) has a $90^o$-phase difference with the excitation force.

Back injuries, especially lower back pain, are amongst the most prevalent and costly non-lethal medical conditions affecting adults. An association between vehicle whole body vibration exposure and the development of lower back pain has been established through numerous epidemiological studies.

In this paper we have presented a concept which could absorb the vibrations that appear at the driver or the passenger of a car or bus. To protect and prevent the lower back pains generated by the car vibrations and the rough road we construct a vibroisolator that can be inserted in the seats.


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