5th International Conference
ðAdvanced Composite Materials Engineering ð
COMAT 2014
16-17 October 2014, Braşov, Romania

STEADY PRECESSION OF A ROLLING DISK OR RING

G.Deliu, D.Botezatu
Transilvania University, Braşov, ROMANIA, deliumec70@gmail.com
Transilvania University, Braşov, ROMANIA, dan.botezatu@gmail.com

Abstract): *The paper presents the deduction of motion equations for any circular body moving in a turn, with application to the particular case of a monocycle vehicle. In this sens, the authors start from the simplest case of steady turning rolling disk. Inspite of the impression of over simplified model, the results of this paper may be largely applied to the study of the motion of an actual monocicle vehicle[1]. The most important conclusion is that such a vehicle is more stable in the turn than some might think.*
Keywords: *steady 1, motion 2, fixed point 3, precession 4, stability 5*

1. INTRODUCTION

The present paper deals with the equations describing a particular case of motion, namely the steady precession of a free rolling body. By steady precession [2], we mean a rotation about a fixed point, having the following particularities: the nutation angle θ = ct., the magnitude of the precession angular velocity $|\dot{\psi}| = |\omega_1|$ = ct., the magnitude of the spin angular velocity $|\dot{\varphi}| = |\omega_0|$ = ct.(Fig.1).

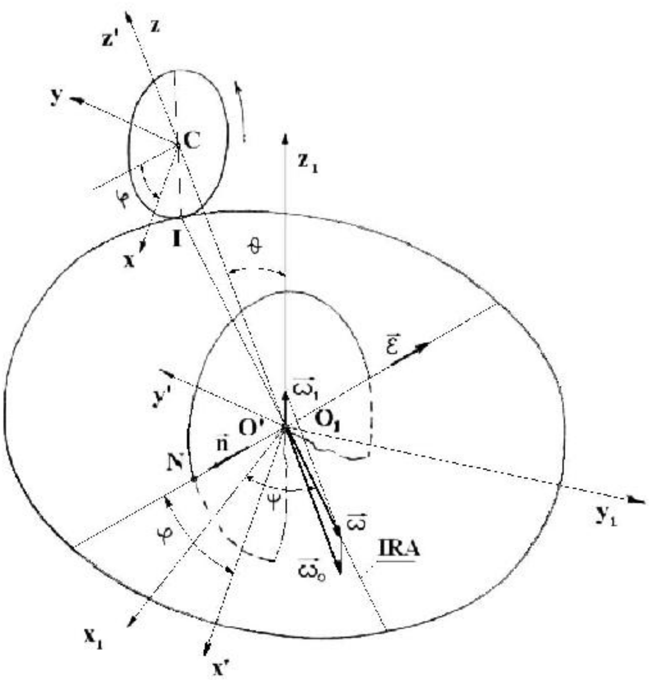

Figure 1. Steady Precession of a Disk

2. REFERENCE FRAMES AND NOTATIONS

To describe the motion of the circular body (disk or ring), we have to use three **reference frames** (Fig.1):

R_1 - *the space frame* $O_1 x_1 y_1 z_1$, fixed to the ground, and having the $O_1 z_1$ – axis vertical upward;

R - *the body frame* $C\,x\,y\,z$, bound to the moving body, having the origin C in the centre of mass of the body and the $C\,z$ – axis passing by the fixed point O_1 **;**

R' – *the reflected frame O'x'y'z'* , parallel to the body frame **R**, but having the origin *O'* over to O_1 .

Notations:

I – contact point between disk and ground; IRA – instantaneous rotation axis; φ - own rotation angle (spin angle); ψ - precession angle; θ - nutation angle; $\vec{\omega}$ - instantaneous (resultant) angular velocity vector; $\vec{\omega}_0$ - spin velocity vector; $\vec{\omega}_1$ - precession velocity vector; $\vec{\varepsilon}$ - angular acceleration vector; $\vec{n}$ - unit vector of the line of nodes.

3. KINEMATIC RELATIONS

3.1. Velocities

As it is well known, a motion about a fixed point may be interpreted as a rolling without slippage of two surfaces one over another – the loci of the instantaneous axis of rotation- that is the *Poinssot's cones:* the body cone rolls on the space cone (Fig.2).

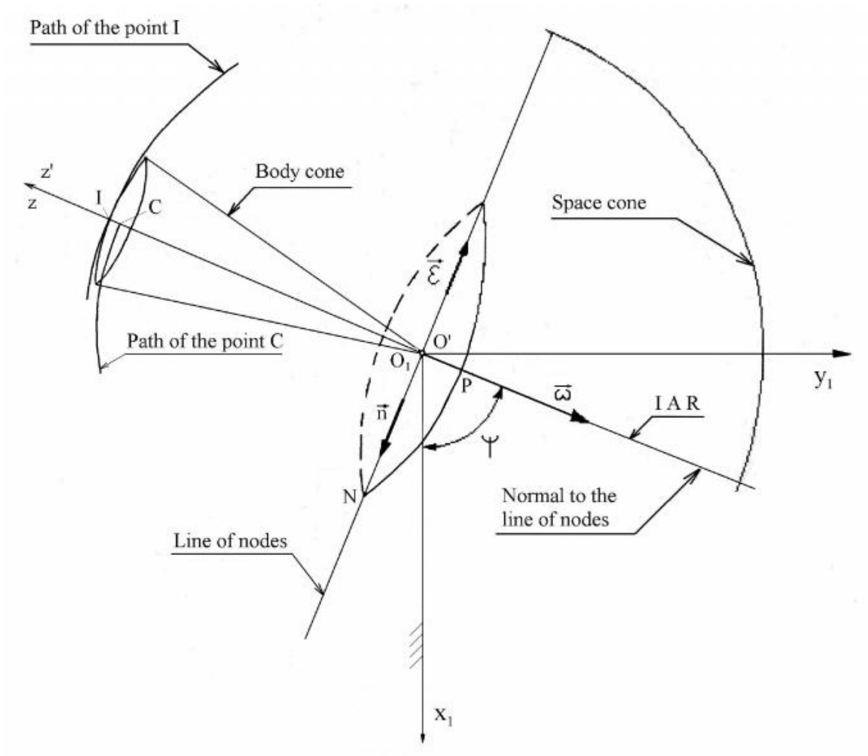

Figure 2. Poinssot's cones

This rotation has the angular velocity (Fig.3)

$$\vec{\omega} = \vec{\omega}_1 + \vec{\omega}_0 \,. \tag{1}$$

The velocity of the point *C* may be written now as

$$\vec{V}_C = \vec{\mu} \times \vec{h} = \left(\vec{\mu}_0 + \vec{\mu}_1\right) \times \vec{h} = \vec{\mu}_1 \times \vec{h}\,. \tag{2}$$

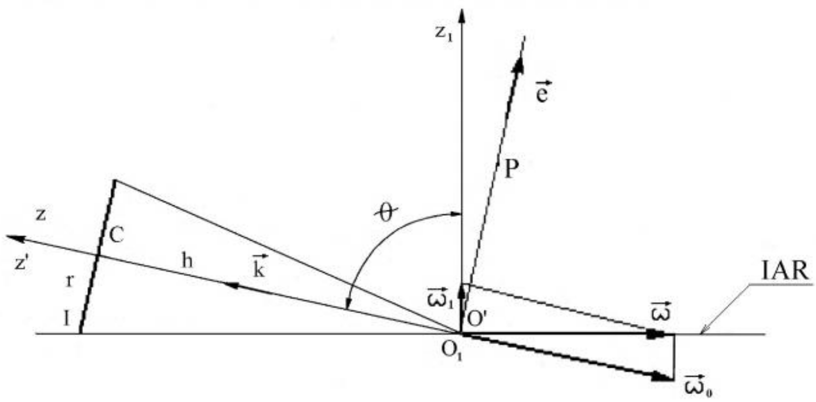

Figure 3. View along the line of nodes

In order to simplify the expressions of the implied vectors, here is useful to introduce another reference frame, namely the *frame of three-orthogonal unit vectors* $\vec{n}, \vec{e}, \vec{k}$ - frame denoted by **R"** (Fig.2 and Fig.3).
In this frame, the velocity of the centre of mass C will have a simpler expression:

$$\left[\vec{V}_C\right]_{R''} = h\mu_1\, sin^{-} \cdot \vec{n}\,, \tag{3}$$

or, in terms of body frame ***R*** components:

$$\left[\vec{V}_C\right]_R = h\mu_1\, sin^{-} \cdot \begin{bmatrix} cos^{\cdot\cdot} \\ sin^{\cdot\cdot} \\ 0 \end{bmatrix}. \tag{3'}$$

Now, we can see (Figure 3), the relations between the magnitudes of angular velocities:

$$\left|\mu_1\right| = \left|\mu_0\right| cos^{-}\,, \tag{4}$$

$$\left|\mu\right| = \left|\mu_0\right| sin^{-}\,. \tag{5}$$

3.2. Accelerations

a) Angular acceleration

From relations above, we can observe that the magnitude of the angular velocity $\vec{\omega}$ is constant. But, because it direction is variable in time, results that the time derivative of the vector $\vec{\omega}$ will be

$$\dot{\vec{\mu}} = \vec{\mu}_1 \times \vec{\mu} = \vec{\mu}_1 \times \vec{\mu}_0\,. \tag{6}$$

Then, we shall obtain:

$$\pounds = \mu_1 \vec{k}_1 \times \mu_0\left(-\vec{k}\right) = -\mu_1 \cdot \mu_0 \cdot sin^{-} \cdot \vec{n}, \tag{7}$$

meaning a vector parallel to the line of nodes and having the magnitude $\left|\pounds\right| = \left|\mu_1\right| \cdot \left|\mu_0\right| \cdot sin^{-}$.

b) Linear acceleration of centre of mass

The acceleration of the centre of mass C can be written starting from the fact that the center of mass has a circular path of radius $R_C = h \cdot sin^{-}$, and that its speed is constant in time. Then, we get

$$\vec{a}_C = \vec{\mu}_1 \times \vec{V}_C\,, \tag{8}$$

or, in terms of the ***R"*** frame components:

$$\vec{a}_C = \begin{vmatrix} \vec{n} & \vec{e} & \vec{k} \\ 0 & \mu_1\, sin^{-} & \mu_1\, cos^{-} \\ h\mu_1\, sin^{-} & 0 & 0 \end{vmatrix}, \qquad \Rightarrow \left[\vec{a}_C\right]_{R''} = h\mu_1^2\, sin^{-} \begin{bmatrix} 0 \\ cos^{-} \\ -sin^{-} \end{bmatrix}. \tag{9}$$

Finally, the magnitude of the acceleration of the centre of mass will be simply

$$\left|\vec{a}_C\right| = h\mu_1^2 \, sin^{-} \; . \tag{10}$$

4. DYNAMICS OF THE STEADY PRECESSION OF A DISK

4.1. Momentum and Angular Momentum about the Centre of Mass

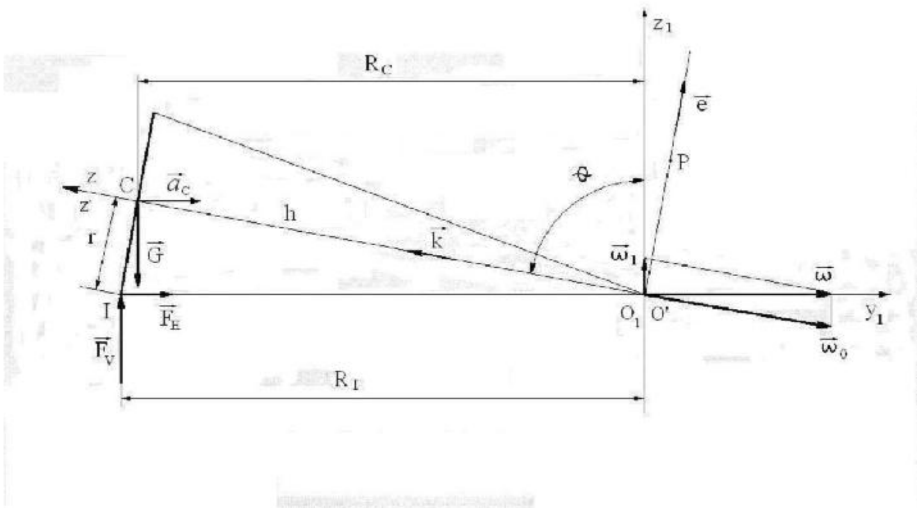

Figure 4. Dynamics of a Steady Precession of a Disk

In order to find the differential equations of motion, we have to write first the expressions of :
a) *Resultant Momentum*,

$\vec{p} = m \cdot \vec{V}_C$, or according to equation (3),

$$\vec{p} = m \cdot h \cdot \mu_1 \, sin^{-} \cdot \vec{n} \; . \tag{11}$$

b) *Angular Momentum about C* :

$$\left[\vec{K}_C\right]_R = \left[\hat{J}_C\right]_R \cdot [\vec{\mu}]_R \, , \tag{12}$$

where the **inertia tensor** about *C* in the frame *R* is

$$\left[\hat{J}_C\right]_R = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} . \tag{13}$$

Then, we shall have

$$\left[\vec{K}_C\right]_R = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} \cdot \begin{bmatrix} \mu_x \\ \mu_y \\ \mu_z \end{bmatrix} = \begin{bmatrix} J_x\mu_x \\ J_y\mu_y \\ J_z\mu_z \end{bmatrix}, \text{ meaning } \left[\vec{K}_C\right]_R = \begin{bmatrix} J_x\mu_1 \, sin^{-} \, sin^{\cdot\cdot} \\ J_y\mu_1 \, sin^{-} \, cos^{\cdot\cdot} \\ J_z\left(\mu_0 + \mu_1 \, cos^{-}\right) \end{bmatrix} . \tag{14}$$

4.2. Time Derivatives

In order to apply the **theorem of momentum variation**, now we must take the time derivatives of the momentum, and respectively of the angular momentum. These will be:

$$\text{a) } \dot{\vec{p}} = m \cdot \vec{a}_C = m\left(\vec{\mu}_1 \times \vec{V}_C\right) . \tag{15}$$

In accordance to equation (9) and Figure 4, we shall obtain in the frame ***R*** the vector

$$. \qquad \left[\dot{\vec{p}}\right]_R = mR_C\mu_1^2 \begin{bmatrix} -cos^{-} \, sin^{\cdot\cdot} \\ cos^{-} \, cos^{\cdot\cdot} \\ -sin^{-} \end{bmatrix} . \tag{16}$$

$$\text{b) } \dot{\vec{K}}_C = \frac{\partial \vec{K}_C}{\partial t} + \vec{\mu} \times \vec{K}_C \, , \tag{17}$$

where $\left[\dfrac{\partial \vec{K}_C}{\partial t}\right] = \left[\hat{J}_C\right] \cdot [\pounds],$ (18)

with

$$[\pounds]_R = -\mu_0 \mu_1 \, sin\text{¯} \begin{bmatrix} cos\,\text{¨} \\ sin\,\text{¨} \\ 0 \end{bmatrix}. \quad (19)$$

Now, the relation (18) becomes

$$\left[\frac{\partial \vec{K}_C}{\partial t}\right] = \begin{bmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{bmatrix} \cdot \begin{bmatrix} -cos\,\text{¨} \\ -sin\,\text{¨} \\ 0 \end{bmatrix} \cdot \mu_0 \mu_1 \, sin\text{¯} = \begin{bmatrix} J_x \pounds_x \\ J_y \pounds_y \\ 0 \end{bmatrix}. \quad (20)$$

Here, the inertia tensor for a disk has the expression $\left[\hat{J}_C\right]_R = \dfrac{mr^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$. (21)

Then, relation (20) takes the form

$$\left[\frac{\partial \vec{K}_C}{\partial t}\right]_R = \frac{mr^2}{4} \mu_0 \mu_1 \, sin\text{¯} \cdot \begin{bmatrix} -cos\,\text{¨} \\ -sin\,\text{¨} \\ 0 \end{bmatrix}. \quad (22)$$

Now, returning to the equation (17), the last term in the right member can be written as

$$\vec{\mu} \times \vec{K}_C = \begin{bmatrix} 0 & -\mu_z & \mu_y \\ \mu_z & 0 & -\mu_x \\ -\mu_y & \mu_x & 0 \end{bmatrix} \cdot \begin{bmatrix} J_x \mu_x \\ J_y \mu_y \\ J_z \mu_z \end{bmatrix} = \begin{bmatrix} \mu_y \mu_z (J_z - J_y) \\ \mu_z \mu_x (J_x - J_z) \\ \mu_x \mu_y (J_y - J_x) \end{bmatrix}. \quad (23)$$

Finally, introducing (20) and (23) in (17), we get:

$$\left[\dot{\vec{K}}_C\right]_R = \begin{bmatrix} J_x \pounds_x + \mu_y \mu_z (J_z - J_y) \\ J_y \pounds_y + \mu_z \mu_x (J_x - J_z) \\ 0 \end{bmatrix}. \quad (24)$$

4.3. External Forces

As we can see from Fig.4, on the disk are acting the following external forces: the weight *G=mg*, the vertical component of the reaction of the ground F_V and the horizontal component of the reaction of the ground F_H.

Note. Rolling resistance is assumed to be negligible in this stage of the study, as well as the tangential to the disk component of the ground reaction.

4.4. Motion Equations

According to the *Theorem of Momentum Variation*, we can write:

$\dot{\vec{p}} = \sum F_i$, that is $m\vec{a}_C = \vec{G} + \vec{F}_V + \vec{F}_H$. (25)

Now, as the acceleration of the centre of mass has no component on the vertical axis, we have the obvious relation

$\vec{G} + \vec{F}_V = \vec{0}$, (26)

which gives further

$m\vec{a}_C = \vec{F}_H$. (27)

Now, in terms of *the angular momentum variation,* we can write: $\dot{\vec{K}}_C = \vec{M}_C$, that is to say

$$\begin{bmatrix} J_x \pounds_x - \mu_y \mu_z (J_y - J_z) \\ J_y \pounds_y - \mu_z \mu_x (J_z - J_x) \\ 0 \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ 0 \end{bmatrix}, \quad (28)$$

where:

$$M_x = \vec{M}_C \cdot \vec{i} = mr\left(a_c \sin\text{¯} - g\cos\text{¯}\right)\cos\text{¨}\ , \qquad (29)$$

$$M_y = \vec{M}_C \cdot \vec{j} = mr\left(a_c \sin\text{¯} - g\cos\text{¯}\right)\sin\text{¨}\ . \qquad (30)$$

Now, introducing in (30) the relations (3') and (19), we get:

$$-J_x \mu_0 \mu_1 \sin\text{¯} \cos\text{¨} + \mu_1 \sin\text{¯} \cos\text{¨} \left(\mu_0 + \mu_1 \cos\text{¯}\right)\left(J_z - J_y\right) = M_C \cos\text{¨}\ , \qquad (31)$$

$$-J_y \mu_0 \mu_1 \sin\text{¯} \sin\text{¨} + \mu_1 \sin\text{¯} \sin\text{¨} \left(\mu_0 + \mu_1 \cos\text{¯}\right)\left(J_z - J_x\right) = M_C \sin\text{¨}\ . \qquad (32)$$

The most important result of these relations is: the resultant moment of the external forces is constant in magnitude and is pointing along a parallel to the line of nodes. As we can easily see after some simple calculi, the magnitude of this moment is for a rolling disk:

$$M_C = J_x \mu_1^2 \sin\text{¯} \cos\text{¯} = \frac{mr^2}{4}\mu_1^2 \sin\text{¯} \cos\text{¯}\ . \qquad (33)$$

On the other hand, from the relations (29) and (30), we can write the same moment as

$$M_C = mr\left(a_C \sin\text{¯} - g\cos\text{¯}\right). \qquad (34)$$

Equalizing these values, in accordance with the *angular momentum variation theorem*, we get the equilibrium relationship equation between the precession speed μ_1 and the nutation angle θ :

$$tg^{2}\text{¯} = 1 + \frac{g}{r\mu_1^2}\ . \qquad (35)$$

3. CONCLUSION

1. Under the conditions of steady precession motion, the lateral inclination of the rolling body (disk or ring) will be constant thanks to inertial (gyroscopic) properties of the motion.

2. The above result is valid also for a monocycle in steady turning, under good adherence conditions, and with accurate keeping of the centre of mass in the symmetry plane of the vehicle,after the turn was started.

REFERENCES

[1] Deliu G.,Deliu M., Rolling Disk Dynamics, *RECENT*, Vol.10,nr.3(27), Braşov, November, 2009

[2] Deliu G., Mecanica, Editura Albastră, Cluj-Napoca, 2003