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ASPECTS CONCERNING THE MODAL ANALYSIS OF THE PLATES MADE OF SYMMETRIC LAMINATED COMPOSITE MATERIAL

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Abstract: *The paper shows some theoretical aspects concerning the analytical modelling of the dynamical behaviour of a composite plate (free vibrations, composite plate with clamped edges). To simplify the mathematical model, only the case of the plate made of a symmetric laminated composite material is considered. Beginning from the equation of motion, the modal analysis consists in seeking of both natural frequencies and natural mode shapes of vibrations to completely describe the dynamical behaviour of such a composite plate. Then, the model is used to analyse the dynamical behaviour of a plate made of an epoxy resin unidirectional reinforced with continuous E-glass fibres. From boundary conditions point of view, the work analyses only the case of all edges embedded. Finally, the results obtained in case of the particular plate involved, will be compared with the results obtained by finite element analysis (FEA).The small values of the error shows the accuracy of the numerical model proposed to analyse dynamical behaviour of any plate made of laminated composite material.*
Keywords: *modal analysis, natural frequencies, composite material*

1. INTRODUCTION

This paper focus on some aspects concerning both of the analytical modelling and numerical simulation (FEA) of the dynamical behaviour of a composite plate (free vibrations, composite plate with clamped edges).The *Rayleigh*'s approximation method proposed in scientific literature [3] is used to compute natural frequencies in case of a composite laminated plate whose all edges are clamped. The main objective of this work is to compare the natural frequencies computed with *Rayleigh*'s approximation with the ones obtained from finite element analysis (FEA).

2. THEORETICAL ASPECTS

It is well-known that the constitutive equation of an element of composite laminated plate [1-4] may be written as follows:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{16} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \\ \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{16} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{16} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix}, \qquad \textbf{(1)}$$

where $\lfloor A_{ij} \rfloor$ is called stretching stiffness matrix; $\lfloor D_{ij} \rfloor$ is bending-twisting stiffness matrix and $\lfloor B_{ij} \rfloor$ is coupling stiffness matrix. The last matrix links the forces developed within the plane of the plate and the curvature vectors. It also links the bending-twisting moments and the displacements within the plane of the plate.
Herein, only the case of the symmetrical laminated composite plate will be studied. This means that some terms of the rigidity matrix ($B_{ij} = 0$; $D_{16} = D_{26} = 0$) vanishes. In this case, the equations of the plate [3] may be written as follows:

$$u_0 = 0; \quad v_0 = 0;$$
$$\left.\begin{array}{l} D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4} + \rho_s \frac{\partial^2 w_0}{\partial t^2} + q = I_{xy}\left(\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + \frac{\partial^4 w_0}{\partial y^2 \partial t^2}\right) \end{array}\right\}, \quad \textbf{(2)}$$

where the quantity ρ_s represents the weight per unit area of the orthotropic layer at point (*x,y*). In the case of a plate of n layers, the layer k having a material density ρ_k, the quantity ρ_s is:

$$\rho_s = \int_{-h/2}^{h/2} \rho\, dz = \sum_{k=1}^{n} \rho_k (z_k - z_{k-1}) = \sum_{k=1}^{n} \rho_k t_k \ , \quad \textbf{(3)}$$

where t_k is the thickness of the layer *k*.

The centrifugal moment of inertia I_{xy} may be neglected and the above equation becomes:

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4} + \rho_s \frac{\partial^2 w_0}{\partial t^2} = 0 . \quad \textbf{(4)}$$

The displacement along the direction perpendicular to the plate, Oz direction, may be expressed in complex form:

$$w_0(x,y,t) = w_0(x,y) e^{i\omega t}, \quad \textbf{(5)}$$

where ω represents the angular frequency of the harmonic vibrations. Substituting this expression into the equation (4) leads to:

$$D_{11}\frac{\partial^4 w_0}{\partial x^4} + 2(D_{12}+2D_{66})\frac{\partial^4 w_0}{\partial x^2 \partial y^2} + D_{22}\frac{\partial^4 w_0}{\partial y^4} + \rho_s \omega^2 w_0 = 0 . \quad \textbf{(6)}$$

The boundary conditions of a rectangular plate simply supported along all edges, may be written:

- for $x = 0$ and $x = a$: $\Rightarrow \begin{cases} w_0 = 0; \\ M_x = 0; \end{cases}$ **(7)**

- for $y = 0$ and $y = b$: $\Rightarrow \begin{cases} w_0 = 0; \\ M_y = 0. \end{cases}$ **(8)**

Using the constitutive equation corresponding to a symmetrical composite plate, leads to the conditions described below:

➢ for $x = 0$ and $x = a$:

$$M_x = -D_{11}\frac{\partial^2 w_0}{\partial x^2} - D_{12}\frac{\partial^2 w_0}{\partial y^2} = 0; \quad \textbf{(9)}$$

➢ for $y = 0$ and $y = b$:

$$M_y = -D_{12}\frac{\partial^2 w_0}{\partial x^2} - D_{22}\frac{\partial^2 w_0}{\partial y^2} = 0. \quad \textbf{(10)}$$

This means that a solution of the above equations (9) and (10) should be [3]:

$$w_0(x,y) = C_{mn} \sin m\pi \frac{x}{a} \sin n\pi \frac{y}{b} \quad \textbf{(11)}$$

and substituting this solution into the equation (6) leads to:

$$\left[\frac{m^4\pi^4}{a^4}D_{11} + 2\frac{m^2n^2\pi^4}{a^2b^2}(D_{12}+2D_{66}) + \frac{n^4\pi^4}{b^4}D_{22} - \rho_s\omega^2\right]C_{mn} = 0 . \quad \textbf{(12)}$$

Since $C_{mn} \neq 0$, the coefficient of C_{mn} should be equal to zero. This condition leads to the expression of the natural frequencies [3] of the transverse vibrations:

$$\omega_{mn} = \frac{\pi^2}{a^2}\sqrt{\frac{1}{\rho_s}\left[m^4 D_{11} + 2m^2n^2R^2(D_{12}+2D_{66}) + n^4R^4D_{22}\right]}, \quad \textbf{(13)}$$

where $R = a/b$. The deformed shape of the plate corresponding to the natural frequencies is given by the relation (11).

For $m = n = 1$, the fundamental frequency of the composite laminated plate may be computed by using the following formula:

$$\omega_{11} = \frac{\pi^2}{a^2}\sqrt{\frac{1}{\rho_s}\left[D_{11} + 2R^2\left(D_{12} + 2D_{66}\right) + R^4 D_{22}\right]}. \qquad \textbf{(14)}$$

The method described above is valid only for the case of a composite plate whose edges are simply supported. In case of the others boundary conditions only approximation methods may be used. Herein, the Rayleigh's approximation is used to compute the natural frequencies.

The *Rayleigh*'s approximation [3] of the vibration frequency of mode *mn* may be written in the following form:

$$\omega_{mn} = \frac{1}{a^2}\sqrt{\frac{D_{11}}{\rho_s}}\sqrt{c_1^4 + 2\left(\alpha_{12} + 2\alpha_{66}\right)R^2 c_2 + \alpha_{22}R^4 c_3^4}\ , \qquad \textbf{(15)}$$

where the coefficients α_{12}, α_{66} and α_{22} are computed with the following formula:

$$\alpha_{12} = \frac{D_{12}}{D_{11}},\quad \alpha_{66} = \frac{D_{66}}{D_{11}},\ \alpha_{22} = \frac{D_{22}}{D_{11}} \qquad \textbf{(16)}$$

while the coefficients c_1, c_2, c_3 are shown in the Table 1 for the case of a composite plate whose edges are clamped.

The natural frequencies ω_{mn} computed with the above relation, is expressed in [rad/s].

Table 1: Coefficients c_1, c_2, c_3 for natural frequencies ω_{mn} of an orthotropic composite plate [3]

m	n	c_1	c_2	c_3
1	1	4.730	12.3^2=151.290	4.730
1	2,3,4,...	4.730	$12.3 \cdot c_3(c_3-2)$	$(n+0.5)\pi$
2,3,4,...	1	$(m+0.5)\pi$	$12.3 \cdot c_1(c_1-2)$	4.730
2,3,4,...	2,3,4,...	$(m+0.5)\pi$	$c_1(c_1-2)c_3(c_3-2)$	$(n+0.5)\pi$

3. DYNAMICAL BEHAVIOUR OF A RECTANGULAR COMPOSITE PLATE WITH CLAMPED EDGES

Herein, it is studied the vibration behaviour of a laminated composite plate $[0 / 90 / 0]_s$ having the dimensions a = 800 *mm*, b = 400 *mm* (Figure 1) while the total thickness of the plate is h = 4.8 *mm*. All layers have the same thickness. Each lamina is made of epoxy resin unidirectional reinforced with continuous *E*-glass fibres. The characteristics of the lamina are $E_1 = 140 \cdot 103$ *MPa*; $E_2 = 5 \cdot 103$ *MPa*; $G_{12} = 5 \cdot 103$ *MPa*; ν_{12} = 0,35; σ_{1t} = 1200 N/mm^2; σ_{1c} = 1000 N/mm^2; σ_{2t} = 50 N/mm^2; σ_{2c} = 120 N/mm^2. Density of the element of volume corresponding to the composite layer is $\rho = 0.0025\, g / mm^3$.

The Figure 2 shows the stacking of the plies of the composite plate analysed.

The first of all, the mathematical model described in the previous section, may be used to compute the natural frequencies corresponding to the first 10 modes of vibration.

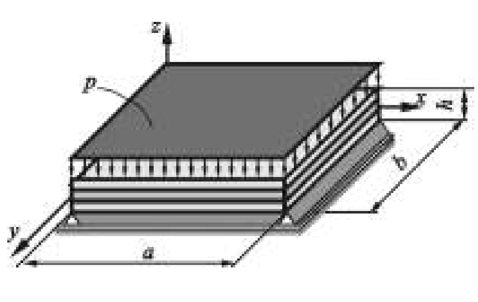


Figure 1: The cases of loading analysed

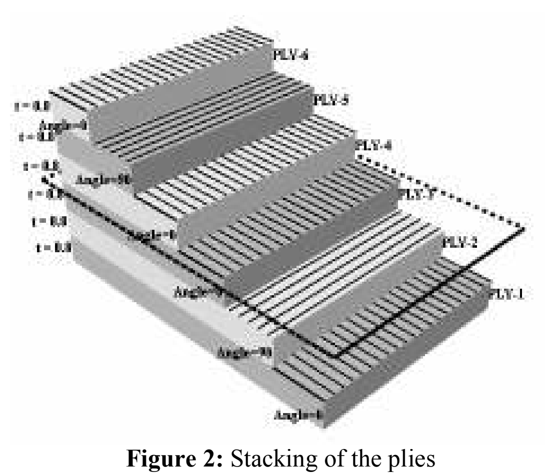
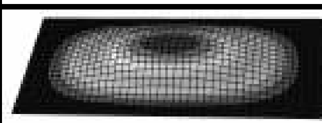
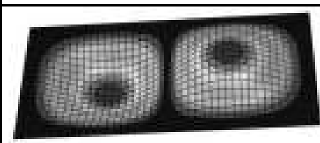
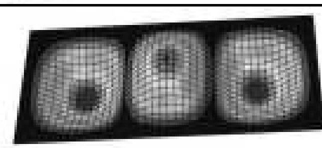
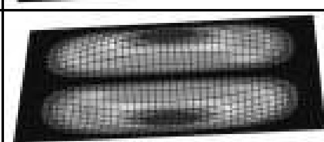
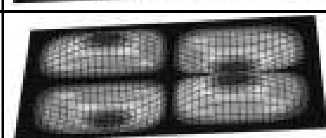
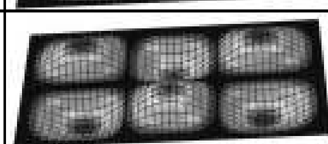
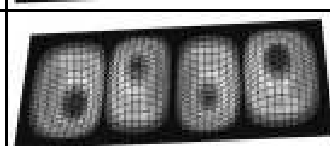
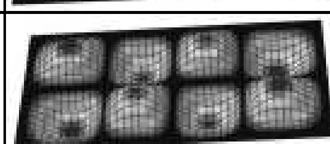
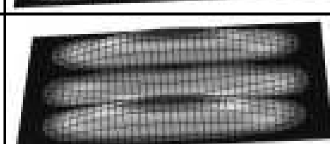


Figure 2: Stacking of the plies

In this particular case, the rigidity matrix corresponding to the composite plate element may be rapidly computed by using a computer program developed by using *MatLab* software within a previous published paper [5]:

Table 2: Natural frequencies and vibration modes in case of a composite laminated plate with clamped edges

No. of vibration mode	m	n	*Rayleigh*'s approximation		Vibration mode	Finite element analysis (FEA)		Error δ (%)
			k_{mn}	Natural frequency ω'_{mn} (cycles / times)		Natural frequency ω''_{mn} (cycles / times)	Eigen value	
1	1	1	60.716	0.13588		0.13619	0.73228	0.22
2	2	1	85.793	0.19201		0.19204	1.4560	0.02
3	3	1	136.944	0.30648		0.30674	3.7145	0.08
4	1	2	155.556	0.34814		0.35184	4.8870	1.06
5	2	2	169.950	0.38036		0.38311	5.7943	0.72
6	3	2	204.613	0.45793		0.45942	8.3327	0.33
7	4	1	211.842	0.47411		0.47567	8.9323	0.33
8	4	2	264.902	0.59286		0.59370	13.915	0.14
9	1	3	301.136	0.67395		0.68995	18.793	2.37
10	5	1	308.355	0.69011		0.69483	19.060	0.68

$$\begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} = \begin{bmatrix} 4.5800 & 0.0844 & 0 & 0 & 0 & 0 \\ 0.0844 & 2.4105 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2400 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9.7193 & 0.1620 & 0 \\ 0 & 0 & 0 & 0.1620 & 3.7026 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4608 \end{bmatrix} x10^5 \,(MPa)\,. \quad \textbf{(17)}$$

The coefficients α_{12} , α_{66} , α_{22} are computed by using the relations (16):

$$\alpha_{12} = 0.016668\,;\ \alpha_{66} = 0.047411\,;\alpha_{22} = 0.380953\,. \quad \textbf{(18)}$$

Taking into account the value of density $\rho = 0.0025\,g/mm^3$ corresponding to the element of volume of a composite layer, the density of an element of area of the composite plate may be easily computed:

$$\rho_s = 0.0025 * 6 * 0.8 = 0.012\,g/mm^2\,, \quad \textbf{(19)}$$

because all layers of the composite material involved are made of the same composite material.

The fifth column of the Table 2 contains the values of the natural frequencies ω'_{mn} computed by using *Rayleigh*'s approximation method.

In the second step, the flexural vibrations behaviour of the composite laminated plate involved in this study was obtained by using the finite element analysis (FEA). To this effect a commercial soft was used to model the composite plate by using composite shell elements. Therefore, it was used a 4-node doubly curved thin or thick shell element, with reduced integration, hourglass control, finite membrane strains. Finally, the model contains 800 elements and 861 nodes. The values of the natural frequencies ω''_{mn} obtained by finite element analysis (FEA) are given in the antepenultimate column of the Table 2.

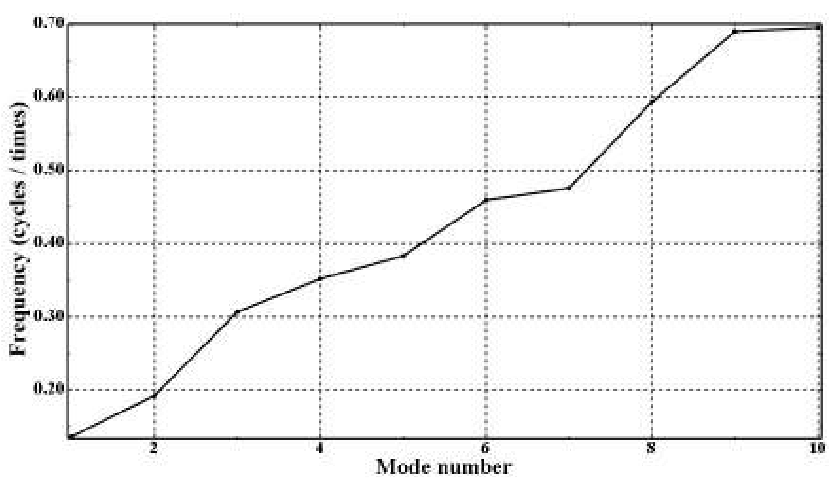


Figure 3: Changing of the natural frequency for each vibration mode in case of the composite plate analysed

The Figure 3 plots the changing of the natural frequency for the first ten vibrations mode.

Finally, the natural frequencies computed by using *Rayleigh*'s approximation method, are compared with the ones obtained by numerical simulation with the method of the finite elements. Thus, the error δ is computed by using the following formula:

$$\delta = \frac{\left|\omega'_{mn} - \omega''_{mn}\right|}{\omega'_{mn}} \quad \textbf{(20)}$$

and the results are reported in the last column of the Table 2 in case of each vibration mode analysed. It may remark that the error δ is acceptable, between 0.02 % and 2.37 %.

4. CONCLUSION

The comparison between the natural frequencies computed by using Rayleigh's approximation and the natural frequencies obtained by finite element analysis (FEA) in case of the numerical example involved shows us the

accuracy of the numerical model used. It follows that this model may easily used to analyse flexural vibrations in case of any composite laminated plate whose edges are clamped.
It may note some important remarks concerning the usefulness of the numerical model developed to analyse the vibration modes and to compute natural frequencies in case of a rectangular composite plate:

- material structure of the composite plate analysed (material corresponding to each layer, number of layers, orientation of the fibres within layers) may be easily changed;
- dimensions of the plate and boundary conditions (all edges simply supported, two edge clamped while the other two edge are free etc) may be easily changed too.

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