Basic results for porous dipolar elastic materials

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Abstract. This paper is concerned with the nonlinear theory of dipolar, porous and elastic solids. Within this theory we obtain, by using the theory of Langenbach, some existence and uniqueness results.

1. Introduction

I.Beju and many other authors have benn establish some existence and uniqueness theorems in the nonlinear classical theory of elastic bodies.

Regarding the polar elastic bodies, we can enumerate the paper [1], where Eringen and Suhubi introduced the theory of microelastic solids. Also, in [2] Eringen developed the theory of micromorphic continua. The theory of multipolar continuum was given by Green and Rivlin in [3]. A review of the field and further developments where recorded by Eringen and Kadafar in [4].

The concept of poros material was introduced by Cowin and Nunziato, in the context of classical theory of Elasticity, in the paper [5]. In this paper and also in the paper [6] of Goodman and Cowin, the authors introduce an additional degree of freedom in order to develope the mechanical behavior of porous solids in which the matrix material is elastic and the interstices are voids material.

The basic premise underling this theory is the concept of a material for which the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field.

The intended applications of this theory are to geological materials like rocks and soil and to manufactured porous materials, like ceramics and pressed powders.

In the present paper we restrict to isothermal processes. Our intention is to extend the result of Beju established in the classical theory of Elasticity. To obtain these results we must make certain assumptions on the material response relating the convexity of internal energy to not be incompatible with the principle of objectivity.

2. Notations and basic equations

Consider that a bounded region B of three-dimensional Euclidian space R^3 is occupied by a poros micropolar body in mechanical equilibrium refered to the reference configuration and a fixed system of rectangular Cartesian axex. Let ∂B be the boudary of the domain B . With $\bar{B}$ we denote the closure of B. Suppose ∂B sufficiently smooth surface such that we can the divergence theorem and Friedrich's inequality.

Letters in bold face stand for vector fields. The components of a vector $\mathbf{v}$ are denoted by v_i The italic indices will always assume the values 1, 2, 3, whereas Greek indices will range over the value 1,2. The Einstein convention regarding the summation on repeated indices is implied. A comma followed by a subscript denotes partial differentiation with respect to the spatial corresponding Cartesian coordinates.

Also, the spatial argument of a function will be omitted when there is no likelihood of confusion.

As usual, we shall denote (X_K) the material coordinates of a typical particle and (x_i) the spatial coordinates of the same particle and we have:

$$x_i = x_i(X_K),\ X_K \in \bar{B}. \tag{1}$$

Suppose the continuous differentiability of the functions x_i with respect to each of the variables X_K, as many times is required. Also, we assume that

$$det\left(\frac{\partial x_i}{\partial X_K}\right) > 0. \tag{2}$$

As in the paper [7] of Truesdell and Noll, we introduce the following additional kinematic variables

$$x_{iA} = x_{iA}(X_K),\ X_K \in \bar{B}, \tag{3}$$

with the property

$$det\,(x_{iA}) \neq 0. \tag{4}$$

This is to characterize the microstructure of the dipolar bodies. To characterize the voids of material we consider that the bulk density ϱ of material is the product of two fields, the density field of matrix material γ and the volum fraction ν, i.e.

$$\varrho = \gamma\nu \tag{5}$$

and this relation also holds for the reference configuration:

$$\varrho_0 = \gamma_0\nu_0.$$

In this way, the deformation of a dipolar body with voids is characterized by the following independent kinematic variables:

$$x_i = x_i(X_K),\ x_{iA} = x_{iA}(X_K),\ \nu = \nu(X_K),\ X_K \in \bar{B}. \tag{6}$$

Using the known procedure of Green and Rivlin, it is easy to obtain that the equations of the equlibrium theory can by written in the following form:

$$\begin{aligned} T_{Ki,K} + \varrho_0 F_i = 0, \\ S_{LiK,L} - S_{iK} + \varrho_0 G_{iK} = 0, \\ H_{K,K} + g + \varrho_0 L = 0. \end{aligned} \tag{7}$$

In these equations we have used the following notations:
- ϱ_0 - the constant mass density (in the reference configuration),
- F_i - the body force,
- G_{iK} - the components of the dipolar body force,
- L - the extrinsic equilibrated body force,
- g - the intrinsic equilibrated body force,
- T_{Ki} - the first Piola-Kirchhoff stress tensor,
- S_{LiK}, S_{iK} - the dipolar stress tensors,
- H_K - the components of the equilibrated stress vector associated with surface in the domain B which where oroginally coordinate planes perpendicular to the X_K- axes through the point (X_K) measured per unit area of these planes.

In the context of the nonlinear theory of dipolar bodies with voids, the consitutive equations are:

$$\begin{gathered} \sigma = \sigma\left(x_{i,K},\ x_{iA},\ x_{iK,K},\ \nu,\ \nu_{,K}\right), \\ T_{Ki} = \frac{\partial \sigma}{\partial x_{i,K}},\ S_{LiK} = \frac{\partial \sigma}{\partial x_{iK,L}},\ S_{iK} = \frac{\partial \sigma}{\partial x_{i,K}},\ H_K = \frac{\partial \sigma}{\partial \nu_{,K}},\ g = \frac{\partial \sigma}{\partial \nu}, \end{gathered} \tag{8}$$

where σ is the internal energy density, considered as a smooth function.
In all what follows, we consider a materially homogeneous body.

We shall denote by u_i and φ_{ij} the components of the displacement field and of the dipolar displacement field, respectively. As it is well known, we have:

$$u_i = x_i - \delta_{iK} X_K,\ \varphi_{ij} = \delta_{iK} x_{jK} - \delta_{iK}\delta_{jN} X_{KN},$$

where δ_{iK} is the Kronecker symbol and X_{KN} is the value of x_{iA} in the reference state.

To the equation of equilibrium (7) we add the following boundary conditions:

$$u_i = \tilde{u}_i,\ \varphi_{ij} = \tilde{\varphi_{ij}},\ \nu = \tilde{\nu}\ \ on\ \ \partial B, \tag{9}$$

where $\tilde{u}_i$, $\tilde{\varphi}_i$ and $\tilde{\nu}$ are prescribed functions.
By summarizing, the boundary-value problem in the context of dipolar bodies with voids consists in finding the functions u_i, φ_{ij} and ν which satisfy the equations (7) and (8) in B and the boundary conditions (9) on ∂B.

3. Basic resuls

In the begining of this section we formulate some results due to Langenbach which will be used in order to obtain the existenxe and uniqueness results in our context.

Consider a bounded domain Ω in n- dimensional Euclidian space R^n. We denote by $\partial\Omega$ the boundary of Ω and consider this surface be sufficiently smooth such that we can apply the divergence theorem. By $\mathcal{H}(\Omega)$ we denote a Hilbert space on Ω.

Let T be an operator $T : D(T) \to \mathcal{H}(\Omega)$, where $D(T) \subset \mathcal{H}(\Omega)$ is the effective domain of the operator T and is a linear subset, dense in $\mathcal{H}(\Omega)$. Suppose that the operator T has a linear Gateaux differential on the set $\omega \subset D(T)$, that is, there exists an operator

$$(DT) : \omega \to L(D(T),\ \mathcal{H}(\Omega)),$$

such that

$$\lim_{t\to 0} \frac{1}{t}\left[T(x+th) - T(x)\right] = (DT)(x)h,\ x \in \omega,\ \ h \in D(T).$$

Here, as usual, we have denoted by $L(D(T),\ \mathcal{H}(\Omega)$ the set of all linear operators defined on $D(T)$, having the values in the Hilbert space $\mathcal{H}(\Omega)$.
The conection between T and (DT) is given by

$$Tx - Tx_0 = \int_0^t (DT)\left(x_0 + \tau\left(x - x_0\right)\right)\left(x - x_0\right) d\tau.$$

We remember that the operator T is called *monotone* if it satifies the relation:

$$\langle Tu - Tv, u - v\rangle \geq 0,\ \forall u, v \in D(T),$$

and T is a *strictly monotone* operator if it is monotone and, in addition, satifies the relation:

$$\langle Tu - Tv, u - v\rangle = 0,\ \Leftrightarrow u = v.$$

Now, we consider the operatorial equation:

$$Tu = f, \tag{10}$$

with the linear and homogeneous boundary-value conditions:

$$L_i u = 0,\ i = \overline{1, m}. \tag{11}$$

Consider the set D_0 defined by:

$$D_0 = \left\{u \in D(T)\ :\ L_i u = 0\right\}.$$

The following theorem allows us to associate a variational problem with our boundary-value problem formulated in Section 2.

Theorem 1. *Consider satisfied the following five conditions:*

- 1) $D_0(T)$ and $D(T)$ are linear sets and $D_0(T)$ is dense in the Hilbert space $\mathcal{H}(\Omega)$;

- 2) the operator T has linear Gateaux differential for all $u,\ h \in D(T)$ and the mapping $(DT)(u)h$ is continuous with respect to u, the value $(DT)(u)h$ belonging to a two-dimensional hyperplane which contains the point u;

- 3) the operator T satisfies the condition:

$$T(0) = 0;$$

- 4) for all $u \in D(T),\ h,\ g \in D_0(T)$, we have:

$$\langle (DT)(u)h,\ g\rangle = \langle (DT)(u)g,\ h\rangle$$

- 5) for all $u \in D(T),\ h \in D_0(T),\ h \neq 0$, we have:

$$\langle (DT)(u)h,\ h\rangle > 0.$$

Then we have:

- i) if there exists a solution $u_0 \in D_0(T)$ of the equation (10), it is unique and attains on $D_0(T)$ the minimum of the functional

$$\Phi(u) = \int_0^t (T(\tau u), u)\, d\tau - (f, u), \tag{12}$$

where $f \in \mathcal{H}(\Omega)$;

- ii) conversely, if an element of $D_0(T)$ attains the minimum of the functional defined in (12), then this element is a solution of the equation (10).

The following theorem has also proved by Langenbach and assures the conditions for the existence and also, for the uniqueness of a generalized solution for the boundary-value problem (10), (11).

Theorem 2. *If we suppose that*

$$\langle (DT)(u)h,\ h\rangle \geq c|h|^2,\ u \in D(T),\ h \in D_0(T),\ c = const.,\ c > 0,$$

then:

- i) the functional (12) is bounded below on $D_0(T)$;

- ii) the functional (12) is strictly convex on $D_0(T)$;

- iii) any minimizing sequence of the functional (12) is convergent in $\mathcal{H}(\Omega)$.

We remember that the limit of a minimizing sequence of the functional (12) is called *generalized solution* of the boundary-value problem (10), (11).
Langenbach has proved that the generalized solution of the problem (10), (11) is unique.

Theorem 3. *We assume that there exists an element $u_0 \in D_0(T)$ such that*

$$\langle (DT)(u)h,\ h\rangle \geq c_1 \langle (DT)(u_0)h,\ h\rangle \geq c_2|h|^2,$$

where c_1 and c_2 are positive constants.
Then the generalized solution of the boundary-value problem (10), (11) is an element of the energetic space of the linear operator $(DT)(u_0)$.

In the following we shall use these theorems to charaterize our above boundary-value problem (7), (8), (9). The equationS (7) can be rewritten in the following form:

$$\begin{aligned} &\left(\frac{\partial\sigma}{\partial u_{i,K}}\right)_{,K} = -\varrho_0 F_i, \\ &\delta_{jK}\left(\frac{\partial\sigma}{\partial\varphi_{ij,L}}\right)_{,L} - \delta_{jK}\frac{\partial\sigma}{\partial\varphi_{ij}} = -\varrho_0 G_{iK}, \\ &\left(\frac{\partial\sigma}{\partial\nu_{,K}}\right)_{,K} + \frac{\partial\sigma}{\partial\nu} = -\varrho_0 L. \end{aligned} \tag{13}$$

The ordered triplets $U = (u_i,\ \varphi_{ij},\ \nu)$ are elements of the real vector space

$$\mathcal{V}_{(13)} = \mathcal{V}_{(3)} \bigoplus \mathcal{V}_{(9)} \bigoplus \mathcal{V}_{(1)}.$$

Of course, the space $\mathcal{V}_{(13)}$ is of 13-dimension and is defined on $\bar{B}$. We now introduce the notations

$$\begin{aligned} M_i\mathbf{U} &= -\left(\frac{\partial\sigma}{\partial u_{i,K}}\right)_{,K}, \\ N_{iK}\mathbf{U} &= -\delta_{jK}\left(\frac{\partial\sigma}{\partial\varphi_{ij,L}}\right)_{,L} - \delta_{jK}\frac{\partial\sigma}{\partial\varphi_{ij}}, \\ P_0\mathbf{U} &= -\left(\frac{\partial\sigma}{\partial\nu_{,K}}\right)_{,K} - \frac{\partial\sigma}{\partial\nu} \end{aligned} \tag{14}$$

and

$$\begin{aligned} M\mathbf{U} &= (M_i\mathbf{U},\ N_{iK}\mathbf{U},\ P_0\mathbf{U}), \\ \mathbf{F} &= (\varrho_0 F_i,\ \varrho_0 G_{iK},\ \varrho_0 L) \end{aligned} \tag{15}$$

Taking into account the notations (14) and (15), the system of equations (13) can be written in the form:

$$M\mathbf{U} = \mathbf{F},\ \textit{on } B. \tag{16}$$

For the sake of simplicity, we denote by $\mathcal{V}$ the space $\mathcal{V}_{(13)}$. Let $\mathbf{V} \in \mathcal{V}$, $\mathbf{V} = (v_i, \psi_{ij}, v)$ such that:

$$v_i = \bar{u}_i,\ \psi_{ij} = \bar{\varphi}_{ij},\ v = \bar{\nu},\ \textit{on } \partial B,$$

where $\bar{u}_i,\ \bar{\varphi}_i$ and $\bar{\nu}$ are prescribed functions defined in (9).
Let us define $\mathbf{W},\ A\mathbf{W}$ and $\mathcal{F}$ by

$$\begin{aligned} \mathbf{W} = \mathbf{U} - \mathbf{V},\ \mathcal{F} &= \mathbf{F} - M\mathbf{V}, \\ A\mathbf{W} = (A_i\mathbf{W},\ B_{iK}\mathbf{W},\ C_0\mathbf{W}) &= M(\mathbf{W}+\mathbf{V}) - M\mathbf{V} \end{aligned} \tag{17}$$

Then the boundary-value problem (7), (8), (9) received the form:

$$A\mathbf{W} = \mathcal{F}, \; on \; B, \tag{18}$$
$$\mathbf{W} = 0, \; on \; \partial B. \tag{19}$$

Let $L_2(B)$ be the Hilbert space of all vector fields $\mathbf{U} = (u_i, \; \varphi_{ij}, \; \nu)$ whose components are square-integrable on B, with the norm generated by the scalar product:

$$\langle \mathbf{U}, \; \mathbf{V} \rangle = \int_B (u_i v_i + \varphi_{ij}\psi_{ij} + \nu v) \, dV$$

where $\mathbf{U} = (u_i, \; \varphi_{ij}, \; \nu)$ and $\mathbf{V} = (v_i, \; \psi_{ij}, \; v)$.
We denote by $W_0^2(B)$ the Sobolev space of all elements from $L_2(B)$ belonging to $C^2(B)$ which satisfy the boundary condition (19). This space will be the domain of definition for the operator A defined in (17), i.e.

$$A : W_0^2(B) \to L_2(B).$$

Also, we suppose that $\mathcal{F} \in L_2(B)$.

Theorem 4. *We assume that the function σ is of class C^2 with respect to each variables $u_{i,K}, \; \varphi_{ij}, \; \varphi_{is,M}, \; \nu, \; \nu_{,N}$ and satisfies the inequality:*

$$\begin{aligned}\Gamma(\mathbf{W}) = \int_B \Big(& \frac{\partial^2 \sigma}{\partial u_{i,K} \partial u_{j,M}} f_{i,K} f_{j,M} + 2 \frac{\partial^2 \sigma}{\partial u_{i,K} \partial \varphi_{sj}} f_{i,K} g_{sj} + \\ & + 2 \frac{\partial^2 \sigma}{\partial u_{i,K} \partial \varphi_{sj,M}} f_{i,K} g_{sj,M} + 2 \frac{\partial^2 \sigma}{\partial u_{i,K} \partial \nu} f_{i,K} h + 2 \frac{\partial^2 \sigma}{\partial u_{i,K} \partial \nu_{,M}} f_{i,K} h_{,M} + \\ & + \frac{\partial^2 \sigma}{\partial \varphi_{is,K} \partial \varphi_{rj,M}} g_{is,K} g_{rj,M} + 2 \frac{\partial^2 \sigma}{\partial \varphi_{is,K} \partial \varphi_{rj}} g_{is,K} g_{rj} + 2 \frac{\partial^2 \sigma}{\partial \varphi_{is,K} \partial \nu} g_{is,K} h + \\ & + 2 \frac{\partial^2 \sigma}{\partial \varphi_{is,K} \partial \nu_{,M}} g_{is,K} h_{,M} + \frac{\partial^2 \sigma}{\partial \varphi_{is} \partial \varphi_{jr}} g_{is} g_{jr} + 2 \frac{\partial^2 \sigma}{\partial \varphi_{is} \partial \nu} g_{is} h + 2 \frac{\partial^2 \sigma}{\partial \varphi_{is} \partial \nu_{,K}} g_{is} h_{,K} + \\ & + \frac{\partial^2 \sigma}{\partial \nu \partial \nu} h^2 + \frac{\partial^2 \sigma}{\partial \nu_{,K} \partial \nu_{,M}} h_{,K} h_{,M} + 2 \frac{\partial^2 \sigma}{\partial \nu \partial \nu_{,K}} h h_{,K} \Big) \, dv > 0, \end{aligned} \tag{20}$$

for all $\mathbf{W} = (u_i, \; \varphi_{ij}, \; \nu)$, $\mathbf{G} = (f_i, \; g_{ij}, \; h)$, $\mathbf{G} \neq 0$ which possess the partial derivatives of first order with respect to the variable X_K.
Then we have:

- i) if there exists a solution $\mathbf{W} \in W_0^2(B)$ of the equation (18), it is unique and attains on $W_0^2(B)$ the minimum of the functional:

$$\Phi(\mathbf{W}) = \int_0^t \langle A(\tau \mathbf{W}, \; \mathbf{W} \rangle \, d\tau - \langle \mathcal{F}, \; \mathbf{W} \rangle \, ; \tag{21}$$

- ii) conversely, if the minimum of the functional (21), on the space $W_0^2(B)$, is attains in an element $\mathbf{W}_0 \in W_0^2(B)$, then this element is a solution of the equation (18).

Proof. The two assertions of the theorem will be proved if we show that the hypotheses of Theorem 1 are satisfied.
So, we have:

- 1) as it is known, $W_0^2(B)$ is a linear set, dense in $L_2(B)$ (see, for instance Minty [12]);

- 2) for all $\mathbf{W},\ \mathbf{G} \in W_0^2(B)$, the operator A defined in (17) has a linear Gateaux differential given by:

$$(DA_i)\,(\mathbf{W})\mathbf{G} = -\left(\frac{\partial^2\sigma}{\partial u_{i,K}\partial u_{j,M}}f_{j,M} + \frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_{js}}g_{js} + \right.$$
$$\left. + \frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_{js,M}}g_{js,M} + \frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu}h + \frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu_{,M}}h_{,M}\right)_{,K},$$

$$(DB_{ij})\,(\mathbf{W})\mathbf{G} = -\left(\frac{\partial^2\sigma}{\partial \varphi_{ij,K}\partial u_{s,M}}f_{s,M} + \frac{\partial^2\sigma}{\partial \varphi_{ij,K}\partial \varphi_{rs}}g_{rs} + \right.$$
$$\left. + \frac{\partial^2\sigma}{\partial \varphi_{ij,K}\partial \varphi_{rs,M}}g_{rs,M} + \frac{\partial^2\sigma}{\partial \varphi_{ij,K}\partial \nu}h + \frac{\partial^2\sigma}{\partial \varphi_{ij,K}\partial \nu_{,M}}h_{,M}\right)_{,K} +$$
$$+ \frac{\partial^2\sigma}{\partial \varphi_{ij}\partial u_{s,K}}f_{s,K} + \frac{\partial^2\sigma}{\partial \varphi_{ij}\partial \varphi_{rs}}g_{rs} + \frac{\partial^2\sigma}{\partial \varphi_{ij}\partial \varphi_{rs,K}}g_{rs,K} + \frac{\partial^2\sigma}{\partial \varphi_{ij}\partial \nu}h + \frac{\partial^2\sigma}{\partial \varphi_{ij}\partial \nu_{,K}}h_{,K},$$

$$(DC_0)\,(\mathbf{W})\mathbf{G} = -\left(\frac{\partial^2\sigma}{\partial \nu_{,K}\partial u_{i,M}}f_{i,M} + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \varphi_{ij}}g_{ij} + \right.$$
$$\left. + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \varphi_{ij,M}}g_{ij,M} + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \nu}h + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \nu_{,M}}h_{,M}\right)_{,K} +$$
$$+ \frac{\partial^2\sigma}{\partial \nu\partial u_{i,K}}f_{i,K} + \frac{\partial^2\sigma}{\partial \nu\partial \varphi_{ij}}g_{ij} + \frac{\partial^2\sigma}{\partial \nu\partial \varphi_{ij,K}}g_{ij,K} + \frac{\partial^2\sigma}{\partial \nu\partial \nu}h + \frac{\partial^2\sigma}{\partial \nu\partial \nu_{,K}}h_{,K}.$$

It is easy to verify that for a given $\mathbf{G}$, the mapping $(DA)(\mathbf{W})\mathbf{G}$ is continuous with respect to $\mathbf{W}$ in every two-dimensional hyperplane which contains the point $\mathbf{W}$;

- 3) this hypothesis is satisfied because from (17) we deduce that $A(0) = 0$;

- 4) for $\mathbf{W},\ \mathbf{G},\ \mathbf{H} \in W_0^2(B)$, provided that $\mathbf{H} = (h_i,\ \chi_{ij},\ \mu)$ possess the partial derivatives of first order with respect to the variable X_K, we get

$$\langle (DA)(\mathbf{W})\mathbf{G},\ \mathbf{H}\rangle = \int_B \left[(DA_i)\,(\mathbf{W})\mathbf{G}h_i + (DB_{ij})\,(\mathbf{W})\mathbf{G}\chi_{ij} + (DC_0)\,(\mathbf{W})\mathbf{G}\mu\right] dV =$$
$$= \int_B \left[(DA_i)\,(\mathbf{W})\mathbf{H}f_i + (DB_{ij})\,(\mathbf{W})\mathbf{H}g_{ij} + (DC_0)\,(\mathbf{W})\mathbf{H}h\right] dV = \qquad (22)$$
$$= \langle (DA)(\mathbf{W})\mathbf{H},\ \mathbf{G}\rangle$$

- 5) Taking into account the inequality (20) and the equality (22), we deduce that:

$$\langle (DA)(\mathbf{W})\mathbf{H},\ \mathbf{H}\rangle > 0,\ \forall \mathbf{W},\ \mathbf{H} \in W_0^2(B),\ \mathbf{H} \neq 0,$$

that is, the last hypothesis of Theorem 1 is satisfied and the dmonstration of the theorem is complete. ■

Theorem 5. *We suppose that the inequality (20) holds. Then the boundary-value problem (7), (8), (9) has at most one solution* $\mathbf{U} \in C^0(B)$.

Proof. The demonstration of this theorem will be based on the following result (see, for instance, the paper [12] of Minty):

If the domain $D(T)$ of the operator T is convex, then a sufficient condition for T to be strictly monotone on $D(T)$ is that the derivative

$$\frac{d}{dt}\left[\langle T(\mathbf{U}+t\mathbf{G}),\ \mathbf{G}\rangle\right]_{t=0}$$

exists and is positive for all $\mathbf{U},\ \mathbf{V} \in D(T),\ \mathbf{G} = \mathbf{V} - \mathbf{V},\ \mathbf{G} \neq 0.$

In view of this result, consider Z the set of all vector fields $\mathbf{U} = (u_i,\ \varphi_{ij},\ \nu)$ that satisfy the boundary conditions (9).
We shall prove that the operator M defined by (14) and (15) is strictly monotone on Z.
Let $\mathbf{U},\ \mathbf{V} \in Z$, $0 \leq t \leq 1$. It is easy to verify that

$$t\mathbf{U} + (1-t)\mathbf{V} \in Z.$$

Then, by using the inequality (20) and the equality (22), we can prove that

$$\frac{d}{dt}\left[\langle M(\mathbf{U}+t\mathbf{G}),\ \mathbf{G}\rangle\right]_{t=0} =$$

$$= \left[\frac{d}{dt}\int_B \left[M_i(\mathbf{U}+t\mathbf{G})f_i + N_{ij}(\mathbf{U}+t\mathbf{G})g_{ij} + P_0(\mathbf{U}+t\mathbf{G})h\right] dV\right]_{t=0} =$$

$$= \langle (DA)(\mathbf{U})\mathbf{G},\ \mathbf{G}\rangle\,,$$

for all $\mathbf{U},\ \mathbf{V} \in Z,\ \mathbf{G} = \mathbf{V} - \mathbf{U},\ \mathbf{G} \neq 0$ *on* ∂B.
Therefore, we deduce that the operator M is strictly monotone on the set Z. As a consequence, if $\mathbf{U}_1$ and $\mathbf{U}_2$ are two solutions of our problem, then by direct calculations we get:

$$\langle M\mathbf{U}_1 - M\mathbf{U}_2,\ \mathbf{U}_1 - \mathbf{U}_2\rangle = \langle 0,\ \mathbf{U}_1 - \mathbf{U}_2\rangle = 0,$$

such that we deduce that $\mathbf{U}_1 = \mathbf{U}_2$, according to the definition of a strictly monotone operator. ■

Following the proof of Theorem 2, we immediately obtain the next results.

Theorem 6. *We suppose that the hypotheses of Theorem 4 are satisfied. Moreover, assume that:*

$$\Gamma(\mathbf{W}) > c\int_B \left(f_i f_i + g_{ij}g_{ij} + h^2\right) dV, \tag{23}$$

for $\mathbf{W} = (u_i,\ \varphi_{ij},\ \nu)$, $\mathbf{G} = (f_i,\ g_{ij},\ h)$ *having the partial derivatives of first order with respect to the variable* X_K*, and* $c =$ *constant,* $c > 0$*.*
Then we have:

a) the functional (21) is bounded below on $W_0^2(B)$*;*

b) the functional (21) is strictly convex on $W_0^2(B)$*;*

c) any minimizing sequence of the functional (21) is convergent in $L_2(B)$ *and the limit is generalized solution of the problem (18), (19);*

d) this generalized solution is unique.

Regarding to the inequality (23) we make the following observations. Suppose there exists a positive constant c_1 such that for all $\mathbf{W} = (u_i,\ \varphi_{ij},\ \nu)$, $\mathbf{G} = (f_i,\ g_{ij},\ h) \in W_0^2(B)$, we have

$$\begin{aligned}
&\frac{\partial^2\sigma}{\partial u_{i,K}\partial u_{j,M}} f_{i,K} f_{j,M} + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_{js}}{}_{i,K} g_j + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \varphi_{j,M}} f_{i,K} g_{j,M} + \\
&+2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu} f_{i,K} h + 2\frac{\partial^2\sigma}{\partial u_{i,K}\partial \nu_{,M}} f_{i,K} h_{,M} + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_{j,M}} g_{i,K} g_{j,M} + \\
&+2\frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \varphi_j} g_{i,K} g_j + 2\frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \nu} g_{i,K} h + \frac{\partial^2\sigma}{\partial \varphi_{i,K}\partial \nu_{,M}} g_{i,K} h_{,M} + \\
&+\frac{\partial^2\sigma}{\partial \varphi_i\partial \varphi_j} g_i g_j + 2\frac{\partial^2\sigma}{\partial \varphi_i\partial \nu} g_i h + 2\frac{\partial^2\sigma}{\partial \varphi_i\partial \nu_{,K}} g_i h_{,K} + \\
&+\frac{\partial^2\sigma}{\partial \nu\partial \nu} h^2 + \frac{\partial^2\sigma}{\partial \nu_{,K}\partial \nu_{,M}} h_{,K} h_{,M} + 2\frac{\partial^2\sigma}{\partial \nu\partial \nu_{,K}} h h_{,K} > \\
&> c_1 \left(f_{i,K} f_{i,K} + g_i g_i + g_{i,K} g_{i,K} + h_K h_K + h^2 \right).
\end{aligned} \tag{24}$$

On the other hand, by using the Friedrich's inequality, we deduce that there exists a real constant c_2 such that:

$$\begin{aligned}
&\int_B \left(f_{i,K} f_{i,K} + g_{ij} g_{ij} + g_{ij,K} g_{ij,K} + h_K h_K + h^2 \right) dV \geq \\
&\geq c_2 \int_B \left(f_i f_i + g_{ij} g_{ij} + h^2 \right) dV.
\end{aligned} \tag{25}$$

Finally, taking into account the inequalities (24) and (25) we deduce that the condition (23) is satisfied.
As a consequence of Theorem 3, it is easy to obtain the result from the following theorem.

Theorem 7. *We assume that there exists* $\mathbf{W}_0 \in W_0^2(B)$ *and two positive constants* c_1 *and* c_2 *such that:*

$$T(\mathbf{W}) \geq c_1 T(\mathbf{W}_0) \geq c_2 \int_B \left(f_i f_i + g_{ij} g_{ij} + h^2 \right) dV,$$

for all $\mathbf{W},\ \mathbf{G} \in W_0^2(B)$, $\mathbf{G} = (f_i,\ g_{ij},\ h)$*.*
Then the generalized solution of the boundary-value problem (18), (19) belogs to the energetic space of the linear operator $(DA)(\mathbf{W}_0)$*.*

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